HARMONY, CONICS AND PERSPECTIVE

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1. Introduction

Perspective is one of the most highly esteemed achievements of the Renaissance. The artists of that iconic period, focused on how to draw realistically, made important discoveries in how do we perceive three dimensional space, [1]. That knowledge became the cornerstone for the foundation of projective geometry within mathematics, two centuries later. It is attributed to Girard Desargues, who was a coetaneus fellow countryman of Rene Descartes; so that it parallels the dawn of analytic geometry. However, in educational terms, we mathematicians have left that cultural heritage in the realm of the artistic curriculum, and barely exploit it in ours; Desargues, compared to Descartes, is barely known. Why don't we tell kids that *ellipses* are what we see, when we look at a circle? Or, how we should draw it? Teachers prefer the general and ambiguous term *oval* for such curves, because they are not quite sure of this assertion. Furthermore, professional mathematicians may doubt for a minute about it, because there is no standard easy proof of it in our curricular memory. Conics are defined by their metric or algebraic properties, and not as circles in perspective. The concept of *harmony*, may be the path to achieve such a simple understanding of conics; and to appraise with it both the deep cultural interaction of art and mathematics, and the influence of projective geometry within it.

2. Harmony

Defined as a special case of the cross ratio as we still teach it today, harmonicity seems to have been first used by Apolonius related to conics (see e.g. [2]). However, it was observed to be a non metric concept deeply related to perspective until the XIX century by Christian Von Staudt, [7]. It can be defined as follows.

Four concurrent lines form a *harmonic pencil* if there is a parallelogram with two of them as sides, a third as diagonal and the fourth parallel to the other diagonal; the concurrency point is called its *center*; O in Figure 1.a. It is a projective notion because the role played by the line at infinity, may be assumed by any line h not incident with O, called the *horizon*, to define the equivalence relation of *parallelism* as concurrency with h, as in Figure 1.b. A *harmonic range* is the intersection of a harmonic pencil with a line not incident with its center (e.g., the four points in h in Figure 1.b, divided into two pairs coming from "sides" versus "diagonals").

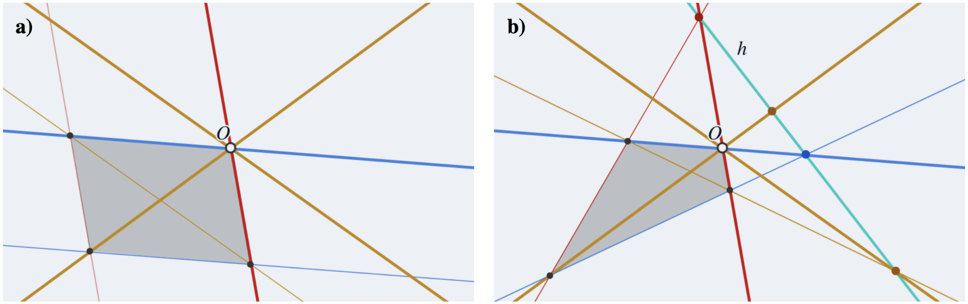

FIGURE 1. **a)** A harmonic pencil with a parallelogram construction. **b)** Parallelism may be defined by any horizon not through the concurrency point.

Dual geometric constructions that yield the *harmonic conjugate* of a point (or line) with respect to a colinear (concurrent) pair of points (lines), follow from the definitions. These constructions are also referred to as the *harmonic fourth*, because they produce a harmonic range (pencil) out of three of its elements, with one distinguished.

Harmonicity is preserved by projections. It is independent of the point of view, in the sense that all points outside the *support* (line) of a harmonic range "see" it as harmonic. However, if four points in general position are considered, some points see them as harmonic but most don't.

Define the *harmonic curve* generated by four points in general position and partitioned into two pairs, to be the locus of points that *see* them as harmonic, that is, which are the center of a harmonic pencil that contains them. There are *generic* points (Figure 2.a), different from the generators, whose lines to them form a harmonic pencil, and the generators which "see themselves" in the harmonic fourth of its lines to the other three (the one in their pair, distinguished); these lines are called the *generating tangents* (Figure 2.b).

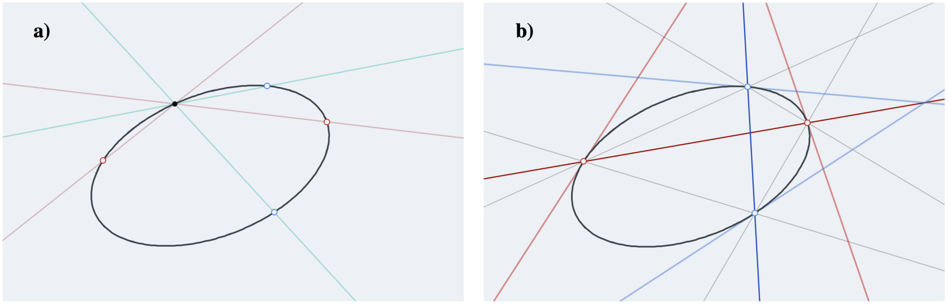

FIGURE 2. **a)** The harmonic curve generated by four points, two red and two blue; **b)** they belong to the harmonic curve because of their tangent lines.

From this, a dual counterpart arises naturally:

The *harmonic bundle* generated by four lines in general position (no three collinear) and partitioned into two pairs, is the set of lines that *see* (or *feel*) them as harmonic, that is, that contain a harmonic range transversal to the lines. Again, the generic lines intersect the generators in a harmonic range, and the generating lines "choose" the harmonic fourth of their intersection with the other three, to "represent" them; it is their *contact point*.

Let us postpone the proof of the natural correspondence between harmonic curves and bundles, to address our main issue.

Theorem 1. *Conics are harmonic curves.*

Proof. First observe that the vertices of a square paired diagonally, generate its circumscribed circle as harmonic curve; it is a high-school exercise (see Figure 3.a). Then, because harmonicity is preserved by projections, the plane section of any cone with a circle as base, is the harmonic curve of the projected generators of the circle (see Figure 3.b). And conics are, by their classic definition, such sections for perpendicular, or straight, cones. □

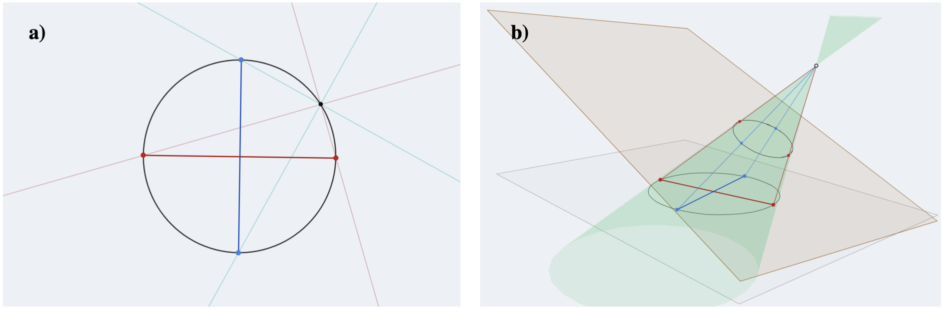

FIGURE 3. **a)** A circle is the harmonic curve generated by the extremes of two perpendicular diameters. **b)** The projection of a harmonic curve is also a harmonic curve; hence, conics are harmonic curves.

A gap in this proof is that it assumes that no other point outside the circle belongs to the harmonic curve. This will be taken care of with the following construction of a harmonic curve.

Given four points in general position and partitioned into the pairs A, B and C, D, let $\mathcal{C}$ denote the harmonic curve they generate, and let ℓ be the line through A and B (see Figure 4.a). Given a point $X \in \ell$, let Y be its harmonic fourth with respect to A, B. Let Z be the intersection of the lines XC and YD. A better notation that we will adopt is

$$Z = (C \vee X) \wedge (D \vee Y)\,; \tag{1}$$

where $\vee$, "join", denotes the linear generating (or closure) operator and $\wedge$, "meet", is intersection. Then, Z sees the generators as harmonic ($Z \in \mathcal{C}$), because its lines to them intersect ℓ in the harmonic range A, X, B, Y (observe that it is written so that the pairs are not consecutive, which is the cyclic order with which they appear on ℓ). As X moves along ℓ, Z traces all of $\mathcal{C}$, because in the line $X \vee C$ there is no point in $\mathcal{C}$, other than Z or C. This implies that $\mathcal{C}$ is bijectively parametrized by the projective line ℓ; or equivalently, by the concurrent pencil of lines centered at C. So that the circumscribed circle is indeed the harmonic curve of the vertices of a square, paired diagonally.

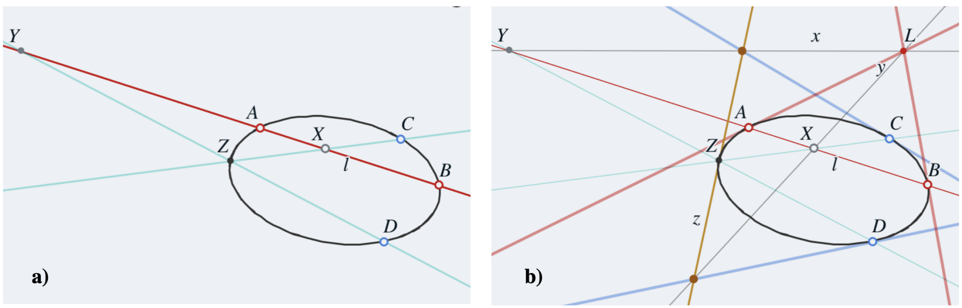

FIGURE 4. **a)** The construction of a harmonic curve. **b)** The dual construction for its harmonic bundle. The point Z and the tangent line through it, z, are the image of D and its tangent line, d, under the harmonic reflection with mirror $y = X \vee L$ and center Y.

With a dynamic geometry system having the harmonic fourth built in as a tool (such as *ProGeo3D*, [6], used to produce the figures), this construction is quite simple. It also derives into insightful examples of parabolas and hyperbolas by sending some of the starting elements of the construction to infinity.

We have proved that circles drawn in perspective are the classic conic curves. Now, the three types fall into where the observer may be. If she is outside the circle, she sees an ellipse; from the inside, he sees a hyperbola, and the parabola (having the line at infinity as a tangent) is the unstable passage from one case to the other.

To see that harmonic curves are paired naturally with harmonic bundles, we need yet another important basic concept associated with harmony. Given a line m and a point $C \notin m$, the *harmonic reflection* with *mirror* m and *center* C, denoted $\rho_{C,m}$, is the map from the projective plane to itself that sends a point to its harmonic fourth with respect to C and the intersection with the mirror m of its line to C.

In other words, consider the harmonic fourth with respect to a pair of points (or with that "mirror pair"), as a map of a projective line to itself, then glue these maps on the pencil of lines about the center C with the other mirror point at the mirror m which thus remains pointwise fixed; close to the mirror it looks as a reflection and close to the center as a central inversion. Of course, harmonic reflections are *colinearities* (send lines to lines), their duals are also harmonic reflections (with roles interchanged) and are *involutions* (they are their own inverse). Furthermore, they can be generalized naturally to 3D with a plane as mirror and a non-incident point as center; and they generalize classic euclidian reflections.

We need a fact about harmonic reflections which we call Klein's Triangle Lemma, because it associates a Klein group (of four elements) to a triangle. It follows directly from the fact that projections preserve harmony; we leave its proof as an exercise.

Lemma 1 (Klein's Triangle)**.** *Given a triangle, the composition of the three harmonic reflections with one vertex as center and the opposite side as mirror is the identity.*

To end this section, let us address the correpondence between harmonic curves and bundles. If $\mathcal{C}$ is the harmonic curve generated by the points A, C, B, D, and a, c, b, d are their respective generating tangents, we will prove that the points in the curve and the lines in the corresponding harmonic bundle are paired by incidence. In Figure 4.b, the dual construction to obtain the bundle is depicted. By Klein's Triangle Lemma on the triangle LXY (where $L = a \wedge b$), with respective opposite sides ℓxy, we have that

$$C \cdot \rho_{X,x} = C \cdot (\rho_{L,\ell} \cdot \rho_{Y,y}) = (C \cdot \rho_{L,\ell}) \cdot \rho_{Y,y} = D \cdot \rho_{Y,y} \,,$$

where we act and compose on the right, and denote both with "$\cdot$". Therefore, Z can also be expressed as $Z = C \cdot \rho_{X,x} = D \cdot \rho_{Y,y}$ because it was defined in (1) as the intersection of lines from the center of the harmonic reflection to the point acted upon, on both expressions. The dual expressions for the line

$$z = c \cdot \rho_{X,x} = d \cdot \rho_{Y,y} = (x \wedge c) \vee (y \wedge d) \,,$$

yield that for each point ($Z = C \cdot \rho_{X,x}$) in the harmonic curve $\mathcal{C}$, there is an incident line ($z = c \cdot \rho_{X,x}$) in the harmonic bundle generated by a, c, b, d, because they are the image of the incident pair $C \in c$ under the harmonic reflection $\rho_{X,x}$.

3. Polarities, ruled surfaces and Dandelin's configuration

The first to give an intrinsically projective definition of conics was, again, Christian Von Staudt in [7]. It is the one Coxeter uses in his influential book [3], and calls it "most appealing" because it has duality built into the definition. It relies on the notion of a *polarity*: a pairing of points and lines (the terms *pole* and *polar* are used) that preserves incidence (or reverses inclusion)[1]. A polarity may be *hyperbolic* if there exists a point incident to its polar line; otherwise it is called *euclidian* (no line contains its pole); and the terms refer to the type of geometry that can be associated to them. The alternative definition of conic curve is then "the points incident with their polar line in a hyperbolic polarity". But in our present approach, we must state it as a theorem:

Theorem 2 (Von Staudt's Polarity Theorem)**.** *A harmonic curve $\mathcal{C}$ induces a polarity (expressed by upper and lower case of the same letter) satisfying:*

i) $P \in \mathcal{C} \Leftrightarrow P \in p$.

ii) *If $P \notin \mathcal{C}$ then the harmonic reflection $\rho_{P,p}$, with P as center and its non-incident polar line p as mirror, leaves $\mathcal{C}$ invariant.*

We have already seen (i) as the pairing by incidence between a harmonic curve and the harmonic bundle associated to its generating tangents. Then, by the preservation of incidence, one can extend the polarity to the rest of the points and lines in the plane. But we will do it with a more insightful and powerful approach. For the moment, observe that

[1]An extra hypothesis is used in [3]; namely that for some line, the pairing of its point range to the line pencil of its pole be a projectivity. But we do not need to stress this issue.

lines and points in Figure 4.b have been named according to the upper and lower case rule for poles and polars with respect to $\mathcal{C}$. In particular, and in view of (ii) in the theorem, that construction gives that the curve is obtained by applying to a fixed point (C) a one parameter family of "symmetries" of the curve $\mathcal{C}$ ($\rho_{X,x}$ with $X \in (A \vee B) \setminus \{A, B\}$), and closing it with A and B.

In terms of the polarity induced by a harmonic curve, the point quadruples that generate it are those for which the pole of the line of one pair is contained in the other line (in Figure 4.b: $(c \wedge d) \in (A \vee B) = \ell$ and thus, $L = (a \wedge b) \in (C \vee D)$). They could rightfully be called *paired harmonic quartets* in $\mathcal{C}$; and in the Klein-Beltrami model of the hyperbolic plane, they correspond to extremes or "points at infinity" of perpendicular lines. In the circle, they are called "cyclic quadrangles" (?).

Our global approach to the polarity theorem is inspired by Germinal Pierre Dandelin's proof to Pascal's Hexagon Theorem, [4]. His idea is to use a three dimensional configuration of six lines associated to a hexagon on a conic curve. Thus, viewed from today, it is most convenient to first review Hilbert and Cohn-Vossen's construction of ruled surfaces, which appeared in print almost a century later, [5].

Consider two lines, say a and b, in three dimensional projective space. They touch if and only if they are coplanar. If this is not the case, we call them a *generating* pair because for any point X not in them, there is a unique line through X *transversal* (i.e., with a common point) to a and b; namely,

$$(X \vee a) \wedge (X \vee b)\,.$$

Now consider three lines a, b, c in *general position* (i.e., each pair is generating). The *transversal ruling* to a, b, c is the set of transversal lines to the three of them called *rules*; and any such set of lines will be called a *ruling* (Figure 5.a). If we denote it $\mathcal{R} = \mathcal{R}(a, b, c)$, the above observation implies that $\mathcal{R}$ is parametrized by incidence with the points in any of the three generating lines (through any point in them there pases a unique rule). It will be important to note that, dually, $\mathcal{R}$ is also parametrized by planes containing one of the lines; if we denote planes by greek letters (points are upper case and lines, lower case latin) we have, for example, that

$$\mathcal{R}(a, b, c) = \{\, (b \wedge \alpha) \vee (c \wedge \alpha) \mid a \subset \alpha \,\}\,. \tag{2}$$

Every pair of rules in $\mathcal{R}$ is generating; otherwise, their three transversal lines a, b, c would be coplanar. Thus, for any triplet $a', b', c' \in \mathcal{R}$ we get a transversal rulling $\mathcal{R}' = \mathcal{R}(a', b', c')$ that contains the original three, a, b, c; this ruling is an *extension* of a, b, c to a ruling (Figure 5.b). In the real projective space (the one outlined by Desargues according to our spatial experience and intuition), it is true that there is only one extension to a ruling of three lines in general position. But there is no simple, or elementary, proof of this fact. Therefore, we state it as an axiom that we suppose valid until further notice.

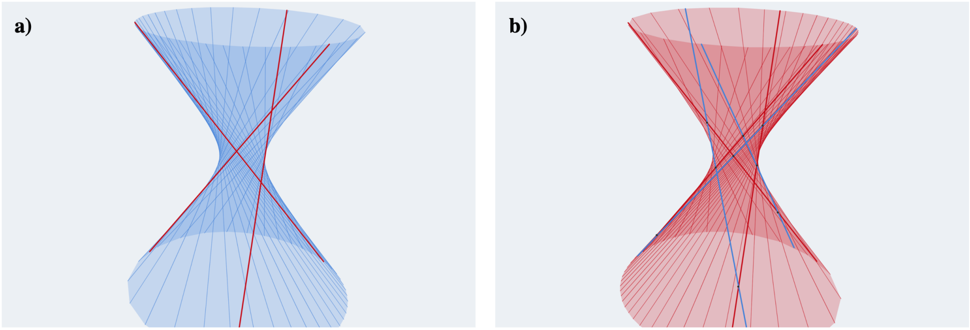

FIGURE 5. **a)** The transversal ruling (in blue) to three red lines. **b)** The transversal ruling to any three blue rules (in red) contains the three red lines.

Axiom of double rulings.[2] *Three lines in general position belong to a unique ruling.*

The name arises because this axiom immediately implies that rulings are paired: any ruling comes with an *opposite ruling* (the transversal ruling to any triplet of its rules). The *ruled surface* (also called *doubly ruled surface*) obtained as the union of the rules in a ruling is also the union of rules in its opposite ruling. Every point on a ruled surface has associated a *tangent* plane: the one generated by the unique rules through it in the

[2]In Spanish, we call this axiom *"Axioma del Equipal"*, refering to a classic, mexican style of furniture that uses double rulings for bases.

two rulings of the surface. They are usually called hyperboloids of one sheet, but also the hyperbolic paraboloids are a special type (the plane at infinity is their tangent).

Now, consider a specific ruling $\mathcal{R}$. It has an opposite ruling $\mathcal{R}'$ and the doubly ruled surface they define is

$$\mathcal{S} = \bigcup_{x \in \mathcal{R}} x = \bigcup_{y \in \mathcal{R}'} y \,.$$

For every point P in $\mathcal{S}$, consider its tangent plane as its *polar* plane. We will prove that this pairing of points in the surface with their tangent planes, extends to a *polarity* throughout space, that is, a pairing between points and planes that preserves incidence. And furthermore, that the harmonic reflection associated to a non-incident polar pair leaves $\mathcal{S}$ invariant (the analogue of Theorem 2).

The basic observation is that any point P not in $\mathcal{S}$, and dually, that any non-tangent plane π, induce natural abstract *matchings* between the two rulings by incidence:

$$\begin{array}{c} \mathcal{R} \overset{(P)}{\longleftrightarrow} \mathcal{R}' \\ x \leftrightarrow y \\ \Updownarrow \\ P \in x \vee y \end{array} \quad , \quad \begin{array}{c} \mathcal{R} \overset{(\pi)}{\longleftrightarrow} \mathcal{R}' \\ x \leftrightarrow y \\ \Updownarrow \\ x \wedge y \in \pi \end{array} \, . \tag{3}$$

So that the polarity induced by $\mathcal{S}$ that we seek is the one where P and π are a polar pair if and only if their corresponding matchings between opposite rulings are identical; and moreover, that this matching is also induced by a global geometric map, in the sense that the harmonic reflection $\rho_{P,\pi}$, with P as center and π as mirror, realizes it.

To prove it, fix a point $P \notin \mathcal{S}$ —we could start, dually, with a non-tangent plane. Consider three rules a, b, c in the ruling $\mathcal{R}$ (beware that we have inverted the notational use of primes: their transversal ruling is now $\mathcal{R}'$). In view of (2) with $\alpha = a \vee P$, there is a well defined rule $a' \in \mathcal{R}'$ for which $P \in a \vee a'$ (it is the match for a under the matching (P) in (3)); let $A = a \wedge a'$. Analougously, we obtain $b', c' \in \mathcal{R}'$, for which $P \in b \vee b' = \beta$ and $P \in c \vee c' = \gamma$; let $B = b \wedge b'$, $C = c \wedge c'$. Our choice of polar plane to P has to be

$$\pi = A \vee B \vee C\,.$$

Now, we show that $\rho_{P,\pi}$ interchanges the lines a, b, c with the corresponding a', b', c' in the opposite ruling. By the triangular symmetry of the construction, it will be enough to prove that:

- *in the tangent plane to A, $\alpha = a \vee a'$, the pair of lines a and a' are harmonic to $A \vee P$ and $\alpha \wedge \pi$;*

because if this is so, the harmonic reflection of the line a with P as center and π as mirror is a'.

We have now distinguished, within the general setting of a doubly ruled surface, what we will call a *Dandelin Configuration*: six lines of two *types* or *colors* —red and blue in the pictures, or unprimed and primed in the text— such that a pair of them touch if and only if they have opposite types. This produces nine *basic* points and nine *tangent* planes by the "wedge" ($\wedge$) or "join" ($\vee$) of lines of different colors; but it also comes with a derived configuration of other lines and planes that naturally arise from them. That combinatorial richness is what Dandelin exploited in [4]; and we follow suit.

The tangent plane $\alpha = a \vee a'$ contains five of the nine basic points of our Dandelin Configuration, namely,

$$a \wedge b', b \wedge a', a \wedge c', c \wedge a'$$

and, of course, the center $A = a \wedge a'$ where the diagonals (a and a') of the previous quadrangle meet. The remaining (outside of α) four basic points, group naturally into two pairs that will relate nicely with the pairs of opposite sides of the above quadrangle.

One pair is $b \wedge c'$ and $c \wedge b'$, whose line $((b \wedge c') \vee (c \wedge b'))$ passes through P, because it is precisely $\beta \wedge \gamma$ (observe that both points lie on both planes), and $\alpha \wedge \beta \wedge \gamma = P$ (see Figure 6.a). Therefore, within α:

$$P = (\alpha \wedge \beta) \wedge (\alpha \wedge \gamma) = ((a \wedge b') \vee (b \wedge a')) \wedge ((a \wedge c') \vee (c \wedge a'))\,.$$

The remaining pair of points is $b \wedge b' = B$ with $c \wedge c' = C$. They are both on the tangent planes $b \vee c'$ and $c \vee b'$ (see Figure 6.b). Therefore, these two planes meet α in $Q = (B \vee C) \wedge \alpha$; which can also be expressed as the intersection of their intersecting lines with α:

$$Q = (\alpha \wedge (b \vee c')) \wedge (\alpha \wedge (c \vee b')) = ((b \wedge a') \vee (a \wedge c')) \wedge ((c \wedge a') \vee (a \wedge b'))\,.$$

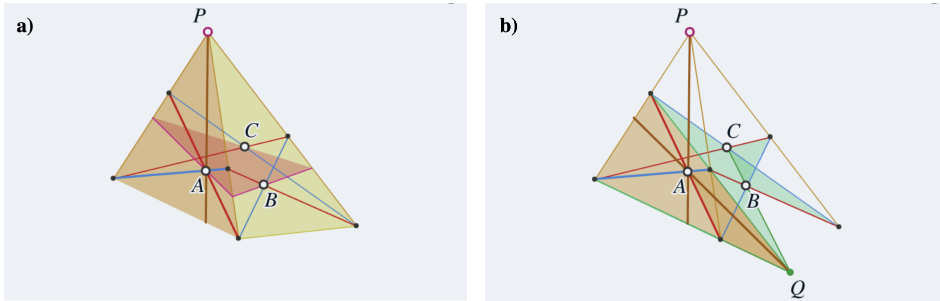

FIGURE 6. **a)** A Dandelin Configuration with a fixed matching of the two types of rules, given by the point P. **b)** The harmonic pencil centered at $A = a \wedge a'$ in the plane $\alpha = a \vee a'$.

The configuration in α that we have described, consisting of 7 points and 8 lines, proves (according to Figure 1.b) what we wanted: a and a' are harmonic with respect to $A \vee P$ and $A \vee Q = \alpha \wedge \pi$.

Thus, $\rho_{P,\pi}$ interchanges the rules a and a'. Analogously, it interchanges b, c with b', c' respectively. Then, it gives a bijection between the transversal rulings of a, b, c and a', b', c', which are $\mathcal{R}'$ and $\mathcal{R}$ respectively; because a line transversal to a, b, c is sent by $\rho_{P,\pi}$ to a line transversal to a', b', c' and viceversa. In particular, since a harmonic reflection sends a line to a line concurrent with the mirror and coplanar with the center, this bijection corresponds to the abstract matchings (3); and it does not depend on our choice of rules a, b, c.

Therefore, $\rho_{P,\pi}$ leaves $\mathcal{S}$ invariant, as we wished to prove.

The proof that the defined pairing between points and planes preserves incidence, and hence is the desired polarity induced by $\mathcal{S}$, is straightforward; we skip it, since its details add little to the picture. Except for the fact that it implies that the polarity extends naturally to a pairing of lines. This pairing has all possible cases: two generating polar lines which do not touch the surface; polar lines which cut the surface in two points, whose tangent planes meet in the polar (many examples were used in the previous paragraphs); tangent pairs of polar lines that meet and form a harmonic pencil with the concurrent rules (an example is $A \vee P$ and $\alpha \wedge \pi$ above); and finally, lines which are self-polar (the rules).

Now, we can turn our attention to the proof of Von Staudt's Polarity Theorem.

Proof (of Theorem 2). Consider, in a plane π, a harmonic curve, $\mathcal{C}$, generated by four points A, C, B, D in general position and paired diagonally. Let a and b be the generating tangents at A and B, respectively; and let $L = a \wedge b$, $\ell = A \vee B$; observe that $D = C \cdot \rho_{L,\ell}$.

Choose a point $P \notin \pi$, to act as pole of π according to a ruled surface $\mathcal{S}$ to be constructed. And finally, choose a third point S in the line $P \vee L$, through which the surface will pass.

Let S' be the harmonic fourth of S with respect to P and L; or equivalently $S' = S \cdot \rho_{P,\pi}$. Since $S \neq S'$, the four lines from S and S' to A and B, can be colored red and blue so that lines of opposite colors touch. And finally, consider the red (blue) line through C transversal to the two blue (red) lines. We now have a Dandelin configuration of six lines colored red and blue: let $\mathcal{S}$ be the doubly ruled surface it defines. By construction, P and π are a polar pair with respect to $\mathcal{S}$.

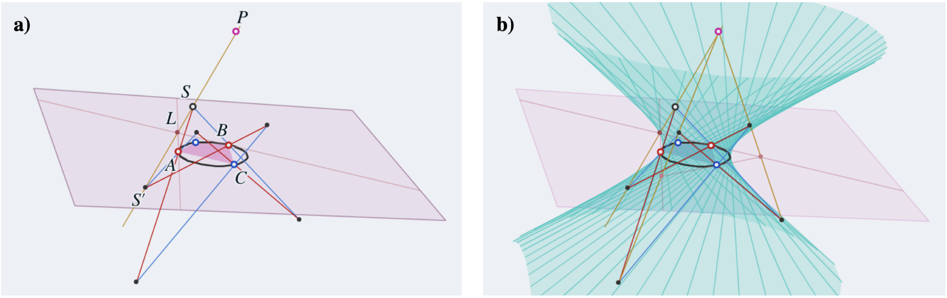

FIGURE 7. **a)** A Dandelin Configuration arising from the generators of a harmonic curve $\mathcal{C}$ in a plane π. **b)** The corresponding ruled surface that intersects π in $\mathcal{C}$.

The polarity induced by $\mathcal{S}$ restricts to a polarity in the plane π: a point has as polar line the intersection of its polar plane with π; and a line has as pole, the pole of the plane it generates with P. Since harmonic reflections preserve the planes through the center, those for non-incident polar pairs with pole in π, preserve the intersection of $\mathcal{S}$ and π. So this intersection has to be $\mathcal{C}$ because, as we saw right after stating Theorem 2, it can be constructed by a family of such harmonic reflections, parametrized by ℓ, and applied to C (or D). $\square$

4. Classic theorems

We have somehow used an alternative definition of a conic curve as the section of a doubly ruled surface with a non-tangent plane. This description was Dandelin's idea to prove Pascal's Hexagon Theorem in [4], considering the alternately colored rules of the surface that an inscribed hexagon in the conic produces. The proof of Pappus Hexagon Theorem, which has the same conclusion than Pascal's, is from certain point on, literally the same; but the plane to be considered is a tangent one. We will detail this case first, for it has the extra corollary of exhibiting the equivalence of the Axiom of Double Rulings with the most classic of projective theorems.

Theorem 3 (Pappus). *Pairs of opposite sides of an hexagon whose vertices lie alternately in two coplanar lines, meet in collinear points.*

Proof. Let $A_1, B_3, A_2, B_1, A_3, B_2$ be the, cyclicly ordered, vertices of such an hexagon (see Figure 9.a). We have named them so that A_1, A_2, A_3 lie in a line, b_0 say, and the vertices B_i lie in a line a_0; let $O = A_0 = B_0 = a_0 \wedge b_0$. If we define,

$$P_i = (A_j \vee B_k) \wedge (A_k \vee B_j)\,, \tag{4}$$

for $\{i, j, k\} = \{1, 2, 3\}$, we must prove that P_1, P_2, P_3 are collinear.

Choose two auxiliary generating lines a_1, a_2 that intersect the plane $\pi = a_0 \vee b_0$ in A_1, A_2 respectively. Let us color a_0, a_1, a_2 red, as opposed to the rules in their transversal ruling, which we color blue. Let b_1, b_2, b_3 be the blue rules through B_1, B_2, B_3 $(\in a_0)$, respectively. Finally, let a_3 be the red line through A_3 transversal to b_1 and b_2. By construction, all opposite colored lines meet except maybe a_3 and b_3; but these two lines meet by the Double Rulings Axiom (one could extend a_0, a_1, a_2 using b_0, b_1, b_3).

Now we have a Dandelin configuration outside of π. And from it, we get, for $\{i, j, k\} = \{1, 2, 3\}$:

$$
\begin{aligned}
P_i &= ((a_j \vee b_k) \wedge \pi) \wedge ((a_k \vee b_j) \wedge \pi) \\
&= ((a_j \vee b_k) \wedge (a_k \vee b_j)) \wedge \pi \\
&= ((a_j \wedge b_j) \vee (a_k \wedge b_k)) \wedge \pi \,.
\end{aligned}
\tag{5}
$$

Therefore (see Figure 9.b), the points P_1, P_2, P_3 lie in the line

$$((a_1 \wedge b_1) \vee (a_2 \wedge b_2) \vee (a_3 \wedge b_3)) \wedge \pi \,.$$

$\square$

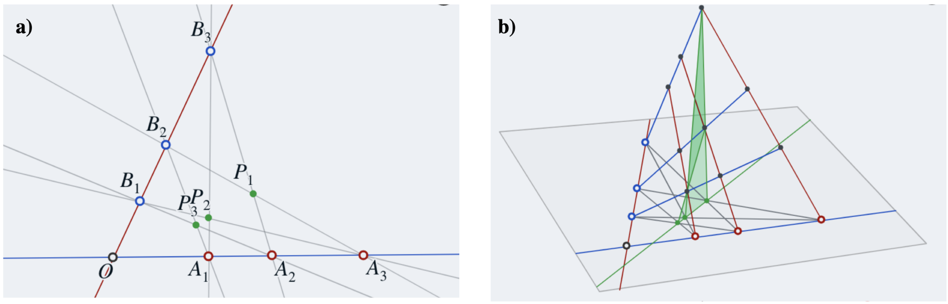

FIGURE 8. **a)** Pappus Theorem and **(b)** Dandelin's proof.

Theorem 4. *The Double Ruling Axiom is equivalent to Pappus Theorem.*

Proof. We have just proved Pappus Theorem using the Double Ruling Axiom. Now assume that Pappus Theorem holds.

The Double Ruling Axiom is clearly equivalent to the following statement: *if four lines belong to a ruling, then a rule in the transversal ruling of three of them also touches the fourth.* That is,

$$b_0, b_1, b_2, b_3 \in \mathcal{R}(a_0, a_1, a_2) \quad \text{and} \quad a_3 \in \mathcal{R}(b_0, b_1, b_2) \quad \Rightarrow \quad a_3 \text{ touches } b_3\,.$$

So lets suppose that a_0, a_1, a_2, a_3 are mutually generating red lines and b_0, b_1, b_2, b_3 mutually generating blue lines, so that all opposite colored pairs meet except for a_3 and b_3; we must prove that they also meet.

Let $A_i = a_i \wedge b_0$ and $B_i = b_i \wedge a_0$, for $i = 1, 2, 3$. By Pappus Theorem in the plane $a_0 \vee b_0$, the three "Pappus points" (4) lie in a "Pappus line" p. The case of (5) that still holds is

$$P_3 = ((a_1 \wedge b_1) \vee (a_2 \wedge b_2)) \wedge \pi \,.$$

Consider the plane $\delta = p \vee (a_1 \wedge b_1) = p \vee (a_2 \wedge b_2)$, with lines $\ell_1 = P_1 \vee (a_1 \wedge b_1)$ and $\ell_2 = P_2 \vee (a_2 \wedge b_2)$. Finally, let $W = \ell_1 \wedge \ell_2$; it is a well defined point because both lines are in the plane δ thanks to Pappus. To see that $W \in a_3$, consider the three planes δ, $a_3 \vee b_1$ and $a_3 \vee b_2$. Using (4), one gets that their pairwise meeting lines are ℓ_1, a_3 and ℓ_2, so that a_3 passes through W. Analogously, W is the meeting point of the planes δ, $a_1 \vee b_3$ and $a_2 \vee b_3$, so that $W \in b_3$; therefore, a_3 meets b_3 and the proof is complete. □

For completeness sake, we briefly reproduce Dandelin's proof of Pascal's Theorem, [4].

Theorem 5 (Pascal)**.** *Pairs of opposite sides of an hexagon whose vertices lie in a harmonic curve, meet in collinear points.*

Proof. Let $A_1, B_3, A_2, B_1, A_3, B_2$ be the vertices of such an hexagon; the cyclic order is according to the hexagon and have nothing to do with the conic curve $\mathcal{C}$ on which they lie. Assuming the Axiom of Doubly Ruled Surfaces —or equivalently, Pappus Theorem— there exists a doubly ruled surface $\mathcal{S}$ that cuts the plane in $\mathcal{C}$. For $i = 1, 2, 3$, let a_i be the rules of one of its rulings that pass through A_i, and b_i the rules in the opposite ruling that contain B_i. The argument now follows verbatim the one we used for Pappus Theorem, once Dandelin's configuration (without a_0 and b_0) is produced. □

A final classic theorem worth mentioning is the following. Where we should stress that the points are now completely general, and not a harmonic quadruple, and hence one more is needed.

Theorem 6. *Through five points in general position in a plane, there passes a unique harmonic curve.*

Proof. Color three of the points red and two blue. Consider two generating blue lines that cut the plane in the blue points. The three red lines incident to the red points and transversal to the chosen blue lines, generate a ruled surface that cuts the plane in a harmonic curve containing the five points.

Uniqueness has two aspects. One is combinatorial. Once there is a chosen surface, for any 3-2 coloring of the points, there is a precise choice of blue lines that yields the same surface. That it does not depend on the chosen surface follows from a construction within the plane based on Pascal's Theorem, whose details are well known. □

5. Axioms for projective geometry

References

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