HARMONY, CONICS AND PERSPECTIVE

JOSE LUIS ABREU AND JAVIER BRACHO

1. Introduction

Perspective is one of the most highly esteemed achievements of the Renaissance. The artists of that iconic period, focusing on how to draw realistically, made important discoveries in the way we perceive three dimensional space, [1]. The knowledge they acquired during this process became, two centuries later, the cornerstone for the creation of projective geometry within mathematics. It is attributed to Girard Desargues, who was a coetaneus fellow countryman of Rene Descartes; so that it parallels the dawn of analytic geometry. However, in educational terms, we mathematicians have left such cultural heritage in the realm of the artistic curriculum and rarely mention it in ours; Desargues, compared to Descartes, is barely known. We define conics by their metric and algebraic properties and ignore the fact that they may be treated entirely within Projective Geometry without even mentioning a metric. The concept of *harmony*, a perfectly projective one, is treated, if at all, as a cross ratio of the measures of certain segments, ignoring also its projective non-metric essence. A purely projective approach for developing the subject of conics, based in the concept of harmony, may be the path for achieving a direct, profound and unified understanding of conics and Projective Geometry; and to appraise with it both the deep cultural interaction between art and mathematics, and the position of projective geometry within it. The goal of this paper is to offer a brief but thorough presentation of such approach.

2. Harmony

Defined as a special case of the cross ratio, as we still teach it today, harmony seems to have been used first in relation to to conics by Apolonius (see e.g. [2]). However, it was not realized to be a non metric concept, deeply related to perspective, before the XIX century, when Karl Georg Christian von Staudt [7] defined it essentially as follows.

Four concurrent lines form a *harmonic pencil* if there is a parallelogram with two of them as sides, a third as diagonal and the fourth parallel to the other diagonal. The point of concurrency is called its *center*; O in Figure 1.a. It is a projective notion because the role played by the line at infinity, may be assumed by any line h not incident with O, called the *horizon*, so that the equivalence relation of *parallelism* is defined as concurrency at a point in h, as shown in Figure 1.b.

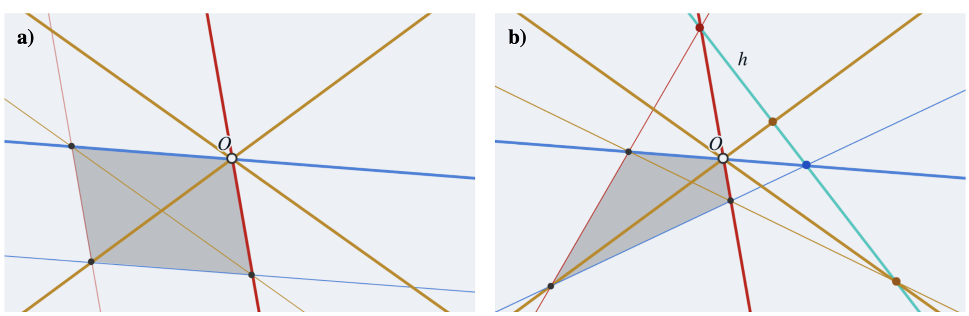

Figure 1. **a)** A harmonic pencil with a parallelogram construction. **b)** Parallelism may be defined by any horizon not through the concurrency point.

A *harmonic range* is formed by the four points of intersections of a harmonic pencil with a line not incident with its center, distinguishing two pairs defined by wether the type of line intersecting h that determines them is a side or a diagonal of the paralelogram (e.g., the two pairs of points on line h in Figure 1.b).

Dual geometric constructions that yield the *harmonic conjugate* of a point (or line) with respect to a colinear (concurrent) pair of points (lines), follow from the definitions. These constructions are also referred to as the *harmonic fourth*, because they produce a harmonic range (pencil) out of three of its elements, with one distinguished from the other two.

Harmony is preserved by projections. It is independent of the point of view, in the sense that all points outside the *support* (line) of a harmonic range "see" it as harmonic (i. e. the four lines joining the outside point to the points on the support line form a harmonic pencil). However, if four points in *general position* (i.e., no three of them being coloinear) are considered, some points "see" them as harmonic but most don't.

We define the *harmonic curve* determined by four points in general position partitioned in two pairs as the locus of points that *see* them as harmonic, that is, which are the center of a harmonic pencil that contains them. We have two kinds of points in such a curve: (1) there are *generic* points, different from the generators, whose lines to them form a harmonic pencil (Figure 2.a); and (2) the generators which "see themselves" in the harmonic fourth of its lines to the other three (the one in their pair, distinguished); these lines are called the *generating tangents* and are paired by incidence with the generating points (Figure 2.b).

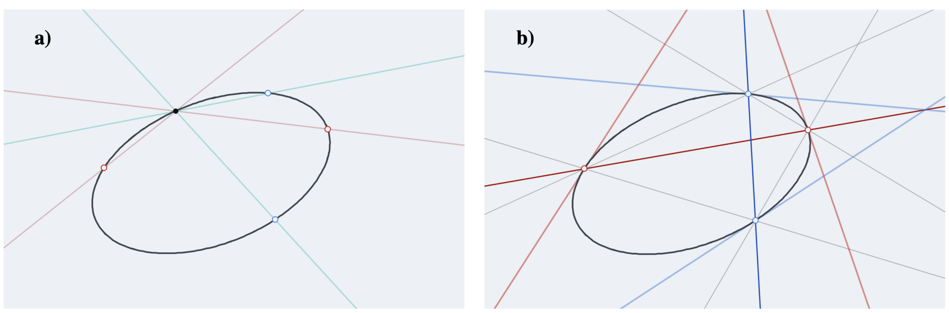

FIGURE 2. **a)** The harmonic curve generated by four points, two red and two blue; **b)** they belong to the harmonic curve because of their tangent lines.

A dual notion arises naturally. The *harmonic bundle* generated (or defined) by four lines in general position (no three collinear) and partitioned into two pairs, is the set of lines that *see* (or *feel*) them as harmonic, that is, that contain a harmonic range *transversal* (i.e., paired by incidence) with the lines. Again, the *generic* lines intersect the generators in a harmonic range, and the generating lines "choose" the harmonic fourth of their intersection with the other three, to "represent" them; it is their *contact point*.

Of course, it is true that each line in the harmonic bundle that arises from the generating tangents of a harmonic curve, passes through a unique point in it (its *contact point* and the line is the *tangent* there). But the proof of this (Theorem 2) needs more work, so we shall address our main issue first.

Theorem 1. *Conics are harmonic curves.*

Proof. First observe that the vertices of a square paired diagonally, generate its circumscribed circle as harmonic curve; it is a high-school exercise (see Figure 3.a). Then, because harmonicity is preserved by projections, the plane section of any cone with a circle as base, is the harmonic curve of the projected generators of the circle (see Figure 3.b). And conics are, by their classic definition, such sections for perpendicular, or straight, cones. ∎

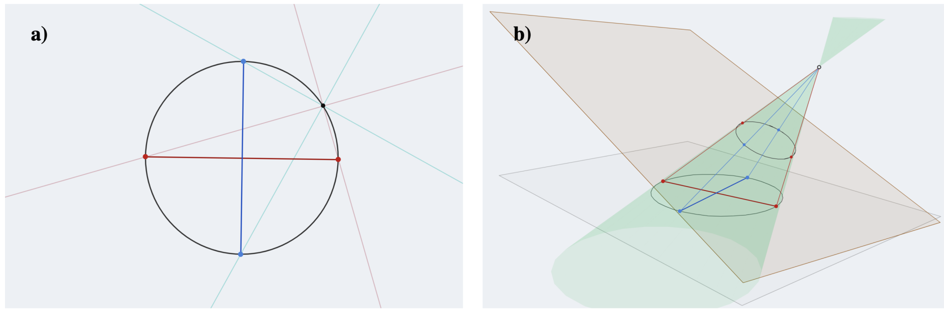

Figure 3. **a)** A circle is the harmonic curve generated by the extremes of two perpendicular diameters. **b)** The projection of a harmonic curve is also a harmonic curve; hence, conics are harmonic curves.

A gap in this proof is that it assumes that no other point outside the circle belongs to the harmonic curve. This will be taken care of with the following construction of a harmonic curve.

Given four points in general position and partitioned into the pairs A, B and C, D, let $\mathcal{C}$ denote the harmonic curve they generate, and let ℓ be the line through A and B (see Figure 4.a). Given a point $X \in \ell$, let Y be its harmonic fourth with respect to A, B. Let Z be the intersection of the lines XC and YD. A better notation that we will adopt is

$$Z = (C \vee X) \wedge (D \vee Y)\,; \tag{1}$$

where $\vee$, "join", denotes the linear generating (or closure) operator and $\wedge$, "meet", is intersection. Then, Z sees the generators as harmonic ($Z \in \mathcal{C}$), because its lines to them intersect ℓ in the harmonic range A, X, B, Y (observe that it is written so that the pairs are not consecutive, which is the cyclic order with which they appear on ℓ).

So that *as X moves along ℓ, Z traces all of $\mathcal{C}$*; because in the line $X \vee C$ there is no point in $\mathcal{C}$, other than Z or C. This implies that $\mathcal{C}$ is bijectively parametrized by the projective line ℓ; or equivalently, by the concurrent pencil of lines centered at C. Therefore, the circumscribed circle is indeed the harmonic curve of the vertices of a square, paired diagonally.

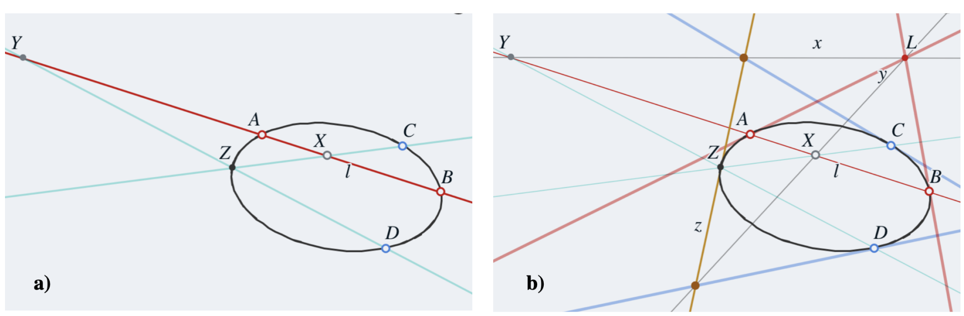

Figure 4. **a)** The construction of a harmonic curve. **b)** The dual construction for its harmonic bundle. The point Z and the tangent line through it, z, are the image of D and its tangent line, d, under the harmonic reflection with mirror $y = X \vee L$ and center Y.

With a dynamic geometry system having the harmonic fourth built in as a tool (such as *ProGeo3D*, [6], used to produce the figures), this construction is quite simple. It also derives into insightful examples of parabolas and hyperbolas by sending some of the starting elements of the construction to infinity.

We have proved that circles drawn in perspective are the classic conic curves. Now, the three types fall into where the observer may be. If she is outside the circle, she sees an ellipse; from the inside, he sees a hyperbola, and the parabola (having the line at infinity as a tangent) is the unstable passage from one case to the other.

To see that harmonic curves are paired naturally with harmonic bundles, we need yet another important basic concept associated with harmony. Given a line m and a point $C \notin m$, the *harmonic reflection* with *mirror* m and *center* C, denoted $\rho_{C,m}$, is the map from the projective plane to itself that sends a point to its harmonic fourth with respect to C and the intersection with the mirror m of its line to C.

In other words, consider the harmonic fourth with respect to a pair of points (or with that "mirror pair"), as a map of a projective line to itself, then glue these maps on the pencil of lines about the center C with the other mirror point at the mirror m which thus remains pointwise fixed; it looks as a reflection close to the mirror and as a central inversion about the center. Of course, harmonic reflections are *colinearities* (send lines to lines), their duals are also harmonic reflections (with roles interchanged) and are *involutions* (they are their own inverse). Furthermore, they can be naturally generalized to 3D with a plane as mirror and a non-incident point as center; and they have as particular cases, classic euclidian reflections and central inversions.

We need a basic fact about harmonic reflections which we call Klein's Triangle Lemma, because it associates a Klein group (of four elements) to a triangle. It follows directly from the fact that projections preserve harmonic ranges; we leave its proof as an exercise.

Lemma 1 (Klein's Triangle)**.** *Given a triangle, the composition of the three harmonic reflections with one vertex as center and the opposite side as mirror is the identity.*

Let us now address the pairing between harmonic curves and harmonic bundles.

Theorem 2 (Duality curves-bundles)**.** *Points in a harmonic curve are* paired *(in bijective correspondence) by incidence with the lines in the harmonic bundle defined by its generating tangents.*

Proof. If $\mathcal{C}$ is the harmonic curve generated by the points A, C, B, D, and a, c, b, d are their respective generating tangents, we will prove that the pairing by incidence between the defining points and lines (expressed as upper and lower case), extends to all the points in the curve and the lines in the corresponding harmonic bundle $\mathcal{C}^*$.

In Figure 4.b, the dual construction to obtain the bundle is depicted. By Klein's Triangle Lemma on the triangle LXY (where $L = a \wedge b$), with respective opposite sides ℓxy, we have that

$$C \cdot \rho_{X,x} = C \cdot (\rho_{L,\ell} \cdot \rho_{Y,y}) = (C \cdot \rho_{L,\ell}) \cdot \rho_{Y,y} = D \cdot \rho_{Y,y}\,,$$

where we act and compose on the right, and denote both with "$\cdot$".

Therefore, for $X \neq A, B$, we can also express Z as $C \cdot \rho_{X,x} = D \cdot \rho_{Y,y}$ because Z was defined in (1) as the intersection of the lines from the center of the harmonic reflection to the point acted upon, on both sides of the equation. The dual expressions for the line

$$z = c \cdot \rho_{X,x} = d \cdot \rho_{Y,y} = (c \wedge x) \vee (d \wedge y)\,,$$

yield that for each generic point, $Z = C \cdot \rho_{X,x}$, in the harmonic curve $\mathcal{C}$, there is an incident line, $z = c \cdot \rho_{X,x}$, called its *tangent*, in the harmonic bundle $\mathcal{C}^*$ because they are the image of the incident pair $C \in c$ under the same harmonic reflection $\rho_{X,x}$. □

As a corollary to the proof we must remark that, with the notation of Figure 4.b used in the preceding theorem, the harmonic curve can also be expressed as

$$\mathcal{C} = \{\, C \cdot \rho_{X,x} \,|\, X \in \ell \setminus \{A, B\} \,\} \cup \{A, B\}\,, \tag{2}$$

where the points A and B must be excluded as parameters of the first set, in order to have the harmonic reflection $\rho_{X,x}$ well defined; but then, they must be added back again to obtain $\mathcal{C}$. Recall that the line x was defined as $x = L \vee Y$, where $L = a \wedge b$ and Y is the harmonic fourth of $X \in \ell = A \vee B$ with respect to A, B; so that all the harmonic reflections, $\rho_{X,x}$, used in (2), interchange the points A and B as well as their generating tangents a and b. They send C to a point in the harmonic curve $\mathcal{C}$. But moreover, we will see that they are symmetries of all of $\mathcal{C}$. The pairing $X \leftrightarrow x$ of points (in ℓ) and lines (through L) is yet another example, other than Theorem 2, of the "polarity" that a harmonic curve induces on the plane. To sate this properly, we need the definition:

A *polarity* is a pairing of points and lines (the terms *pole* and *polar* are used) that preserves incidence (or reverses inclusion)[1].

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[1]An extra hypothesis is used in [3]; namely that for some line, the pairing of its point range to the line pencil of its pole be a projectivity. But we do not need to stress this issue.

Theorem 3 (Polarity)**.** *A harmonic curve $\mathcal{C}$ induces a polarity (expressed by upper and lower case of the same letter) satisfying:*

i) $P \in \mathcal{C} \Leftrightarrow P \in p$.
ii) *If $P \notin \mathcal{C}$ then the harmonic reflection $\rho_{P,p}$, with P as center and its non-incident polar line p as mirror, leaves $\mathcal{C}$ invariant.*

We have already seen item (i) as Theorem 2; tangent lines to a harmonic curve are defined as their polars. The rest of the proof will be given towards the end of the next section; for the moment, let us make three pertinent remarks about the theorem itself.

First, two mathematicians directly associated to this theorem are Jean-Victor Poncelet and Karl G. C. von Staudt. The first, proved the relation of poles and polars of conic sections with harmony, and the second, soon after, developed polarities as a general concept and used it as an alternative way to define conic curves within projective geometry and with no relation to metric or algebraic considerations. This definition is the one Coxeter uses in his influential book [3], and calls it "most appealing" because it has duality built into it. Two types of polarities must be precised: *euclidian* when no point is incident with its polar line, and *hyperbolic* when there exist pole and polar incident pairs; the terms are then related to the groups generated by harmonic reflections on non-incident polar pairs. Von Staudt's definition is equivalent to item (i) of the theorem in a hyperbolic polarity, while Poncelet's results can be rephrased as item (ii).

Second, as examples of polar pairs, we have named lines and points in Figure 4.b according to the upper and lower case rule for poles and polars with respect to the displayed harmonic curve.

Third, in terms of the polarity induced by a harmonic curve, the point quadruples that generate it as such, are those for which the pole of the line of one pair is incident with the other line (in Figure 4.b: $(c \wedge d) \in (A \vee B) = \ell$ and thus, $L = (a \wedge b) \in (C \vee D)$, see also Figure 2.b). They could rightfully be called *harmonic quartets* in $\mathcal{C}$; and in the Klein-Beltrami model of the hyperbolic plane, they correspond to extremes or "points at infinity" of perpendicular lines. In the circle, they are called "cyclic quadrangles" (?).

3. Ruled surfaces

Our approach to prove the Polarity Theorem (3) is inspired by Germinal Pierre Dandelin's proof of Pascal's Hexagonal Theorem, [4]. He considers the plane within three dimensional space and then makes his arguments there. This general idea is classic. It is typified by Desargues' Theorem which becomes almost obvious in three dimensions, and it is quite useful to prove other basic results such as the projective invariance of harmony. It is a natural idea because perspective, as a basic source of projective geometry, makes little sense without three dimensions. Also, conic sections use three dimensional space to be defined in the classic way, but in Dandelin's proof, instead of circular cones, ruled surfaces are used.

We first review Hilbert and Cohn-Vossen's construction of ruled surfaces in [5], which appeared in print almost a century after Dandelin's proof, and made clear that they can be defined depending only on incidence.

Consider two lines, say a and b, in three dimensional projective space. They touch if and only if they are coplanar. If this is not the case, they can be called a *generating* pair because for any point X not in them, there is a unique line through X *transversal* (i.e., with a common point) to a and b; namely,

$$(X \vee a) \wedge (X \vee b)\,,$$

where we are extending the use of the operations $\vee$ and $\wedge$ to all flats.

Now consider three lines a, b, c in *general position* (i.e., each pair is generating). The *transversal ruling* to a, b, c is the set of lines that are transversal to them (i.e., that touch the three); any such set of lines will be called a *ruling* (see Figure 5.a) and its elements are called *rules*. If we denote it $\mathcal{R} = \mathcal{R}(a, b, c)$, the above observation implies that $\mathcal{R}$ is parametrized by incidence with the points in any of the three generating lines (through any point in them there pases a unique rule). It will be important to note that, dually, $\mathcal{R}$ is also parametrized by planes containing one of the lines; if we denote planes by greek letters (points and lines are, respectivelly, upper and lower case latin) we have, for example, that

$$\mathcal{R}(a, b, c) = \{\, (b \wedge \alpha) \vee (c \wedge \alpha) \,|\, a \subset \alpha \,\}\,. \tag{3}$$

Every pair of rules in $\mathcal{R}$ is generating; otherwise, their three transversal lines a, b, c would be coplanar. Thus, for any triplet $a', b', c' \in \mathcal{R}$ we get a transversal rulling $\mathcal{R}' = \mathcal{R}(a', b', c')$ that contains the original three lines, a, b, c; this ruling is an *extension* of a, b, c (see Figure 5.b). In real projective space (the one outlined by Desargues according to our spatial experience and intuition, by incorporating a plane at infinity), it is true that there is only one extension to a ruling of three lines in general position. But there is no simple, or elementary, proof of this fact. Therefore, we state it as an axiom that will be proved to be equivalent to Pappus Theorem and other classic statements that have been used as axioms.

Axiom of double rulings.[2] *Three lines in general position belong to a unique ruling.*

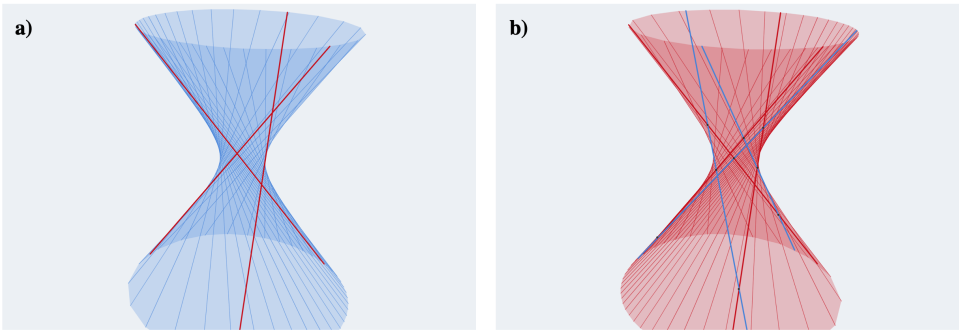

FIGURE 5. **a)** The transversal ruling (in blue) to three red lines. **b)** The transversal ruling to any three blue rules (in red) contains the three red lines.

The name arises because this axiom immediately implies that rulings are paired: any ruling comes with an *opposite ruling* (the transversal ruling to any triplet of its rules). The *ruled surface* (also called *doubly ruled surface*) obtained as the union of the rules in a ruling is also the union of rules in its opposite ruling. They are usually called hyperboloids of one sheet, but also the hyperbolic paraboloids are a special type (they are tangent to the plane at infinity).

Every point on a ruled surface has associated a *tangent* plane: the one generated by the unique rules through it in the two rulings of the surface. If we consider it as its *polar* plane, we will see that this association gives rise to a full scale *polarity*: a pairing between points and planes that preserves incidence. That is, we now prove a three dimensional analogue of the Polarity Theorem (3):

Theorem 4 (Polarity for Ruled Surfaces)**.** *The pairing of points in a ruled surface $\mathcal{S}$ with their respective tangent planes, extends to a polarity of projective space, in which if $P \notin \mathcal{S}$ then P is not incident with its polar plane π, and the harmonic reflection $\rho_{P,\pi}$, with P as center and π as mirror, leaves $\mathcal{S}$ invariant.*

Proof. Consider a specific ruling $\mathcal{R}$. It has an opposite ruling $\mathcal{R}'$ and the doubly ruled surface they define is

$$\mathcal{S} = \bigcup_{x \in \mathcal{R}} x = \bigcup_{y \in \mathcal{R}'} y\,.$$

The basic observation that guides the proof is that any point P not in $\mathcal{S}$ and, dually, that any non-tangent plane π, induce natural abstract bijections (or *matchings*) between the two rulings of $\mathcal{S}$ by incidence:

$$\begin{array}{c} \mathcal{R} \xleftrightarrow{(P)} \mathcal{R}' \\ x \leftrightarrow y \\ \Updownarrow \\ P \in x \vee y \end{array} \quad , \quad \begin{array}{c} \mathcal{R} \xleftrightarrow{(\pi)} \mathcal{R}' \\ x \leftrightarrow y \\ \Updownarrow \\ x \wedge y \in \pi \end{array} \quad . \tag{4}$$

So that the polarity induced by $\mathcal{S}$ that we have to define is the one in which P and π are a polar pair if and only if their corresponding abstract matchings between opposite rulings are identical. And moreover, we have to prove that this matching is also induced by a global geometric map: the harmonic reflection $\rho_{P,\pi}$, with P as center and π as mirror.

To define the polarity induced by $\mathcal{S}$ in its complement, consider a point $P \notin \mathcal{S}$ —we could start, dually, with a non-tangent plane. Fix three rules a, b, c in the ruling $\mathcal{R}$, and beware that we have inverted the notational use of primes: their transversal ruling is now $\mathcal{R}'$.

[2] In Spanish, we call this axiom *"Axioma del Equipal"*, refering to a classic, mexican style of furniture that uses double rulings for bases.

Let $\alpha = a \vee P$. In view of (3), there is a well defined rule $a' \in \mathcal{R}'$ for which $P \in a \vee a' = \alpha$ (namely, $a' = (b \wedge \alpha) \vee (c \wedge \alpha)$ which is the line that corresponds to a under the matching (P) in (4)); let $A = a \wedge a'$. Analougously, we obtain $b', c' \in \mathcal{R}'$, for which $P \in b \vee b' = \beta$ and $P \in c \vee c' = \gamma$; let $B = b \wedge b'$, $C = c \wedge c'$. To have the same matchings (4), the polar plane to P has to be defined as

$$\pi = A \vee B \vee C .$$

Observe that if we had started, dually, with this plane π we would have found P as the intersection of the three tangent planes.

Now, we will show that the harmonic reflection with P as center and π as mirror, $\rho_{P,\pi}$, interchanges the lines a, b, c with the corresponding a', b', c' in the opposite ruling. By the triangular symmetry of the construction, it will be enough to prove that:

- *in the tangent plane to A, $\alpha = a \vee a'$, the pair of lines a and a' are harmonic to $A \vee P$ and $\alpha \wedge \pi$;*

because this happens if and only if $\rho_{P,\pi}$ interchanges the lines a and a'.

We have distinguished, within the general setting of a doubly ruled surface, what we will call a *Dandelin Configuration*: six lines of two *types* or *colors* —red and blue in the pictures, or unprimed and primed in the text— such that a pair of them touch if and only if they have opposite types. This produces nine *basic* points and nine *tangent* planes by the “wedge” ($\wedge$) or “join” ($\vee$) of lines of different colors; but it also comes with a derived configuration of other lines and planes that naturally arise from them. That combinatorial richness is what Dandelin exploited in [4]; and we follow suit.

The tangent plane $\alpha = a \vee a'$ contains five of the nine basic points of our Dandelin Configuration. The *α-quadrangle*:

$$a \wedge b', b \wedge a', a \wedge c', c \wedge a' ,$$

with its center $A = a \wedge a'$ where its diagonals a and a' meet. The remaining four basic points outside of α, group naturally into two pairs which generate concurrent lines with the two pairs of opposite sides of the α-quadrangle.

One pair is $b \wedge c'$ and $c \wedge b'$, whose line $((b \wedge c') \vee (c \wedge b'))$ passes through P, because it is precisely $\beta \wedge \gamma$ (observe that both points lie on both planes), and $\alpha \wedge \beta \wedge \gamma = P$ (see Figure 6.a). Therefore, within α:

$$P = (\alpha \wedge \beta) \wedge (\alpha \wedge \gamma) = ((a \wedge b') \vee (b \wedge a')) \wedge ((a \wedge c') \vee (c \wedge a')) .$$

The remaining pair of points are $b \wedge b' = B$ and $c \wedge c' = C$. They are both on the tangent planes $b \vee c'$ and $c \vee b'$ (see Figure 6.b). Therefore, these two planes meet α in $Q = (B \vee C) \wedge \alpha$; which can also be expressed as the intersection of their intersecting lines with α:

$$Q = (\alpha \wedge (b \vee c')) \wedge (\alpha \wedge (c \vee b')) = ((b \wedge a') \vee (a \wedge c')) \wedge ((c \wedge a') \vee (a \wedge b')) .$$

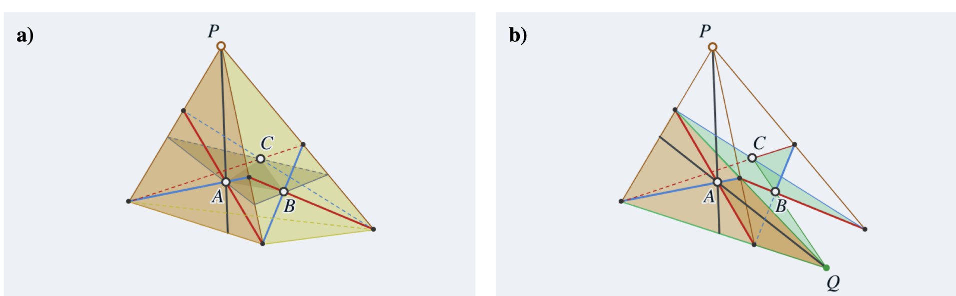

FIGURE 6. **a)** A Dandelin Configuration with a fixed matching of the two types of rules, given by the point P and the plane $\pi = A \vee B \vee C$. **b)** The harmonic pencil $a, A \vee P, a', \alpha \wedge \pi = A \vee Q$, centered at $A = a \wedge a'$ in the plane $\alpha = a \vee a'$.

The configuration in α that we have described, consisting of 7 points and 8 lines, proves (according to Figure 1.b) what we wanted: a and a' are harmonic with respect to $A \vee P$ and $A \vee Q = \alpha \wedge \pi$.

Thus, $\rho_{P,\pi}$ interchanges the rules a and a'. Analogously, it interchanges b, c with b', c' respectively. Then, it gives a bijection between the transversal rulings of a, b, c and a', b', c', which are $\mathcal{R}'$ and $\mathcal{R}$ respectively, because a line transversal to a, b, c is sent by $\rho_{P,\pi}$ to a line transversal to a', b', c' and viceversa. In particular, since a harmonic reflection sends a line to a line concurrent with the mirror and coplanar with the center, this geometric bijection corresponds to the abstract matchings (4); thus, the polar plane π to the point $P \notin \mathcal{S}$ does not depend on our choice of generating rules a, b, c.

Therefore, $\rho_{P,\pi}$ leaves $\mathcal{S}$ invariant, as we wished to prove.

The remaining proof that the pairing we have constructed between points and planes preserves incidence, has to be argued in cases. Let us denote the now well defined *polar plane* of any point X by $X^\perp$.

Let P and Q be two points such that $Q \in P^\perp$, we are left to prove that $P \in Q^\perp$.

The first case is when both P and Q are in $\mathcal{S}$, but then $Q \in P^\perp$ iff they belong to the same rule, because $\mathcal{S} \cap P^\perp$ is the union of the two rules at P. The case when one point is in $\mathcal{S}$ and the other is not, has been dealt with. For we have just used that given $P \notin \mathcal{S}$, the criterion which determines if Q is in $\mathcal{S} \cap P^\perp$ is precisely $P \in Q^\perp$. So we are left with the generic case in which P and Q are not in $\mathcal{S}$.

It is not hard to argue that there exists a line ℓ in the plane $P^\perp$ that passes through Q and contains two different points A, B in $\mathcal{S}$. Let a, a' (and b, b') be the two rules of $\mathcal{S}$ that pass through A (B). Then, $A \in P^\perp$ and $B \in P^\perp$ imply

$$P \in A^\perp \wedge B^\perp = (a \vee a') \wedge (b \vee b') = (a \wedge b') \vee (b \wedge a')\,. \tag{5}$$

Both points in the last expression are in $\mathcal{S}$. Furthermore, we have that $(a \wedge b') \in Q^\perp$ because $Q \in \ell = A \vee B \subset (a \vee b') = (a \wedge b')^\perp$; and likewise $(b \wedge a') \in Q^\perp$. Therefore, (5) implies $P \in Q^\perp$. □

Observe that the polarity extends naturally to a pairing of lines. The polar of a line ℓ is the intersection $\ell^\perp$ of all the polar planes of its points.

This polarity theorem asserts that what one sees as the contour of a ruled surface is exactly its section with the polar plane of the viewpoint. Sections and projections match. We now prove that they are harmonic curves, and if the lines of any of its rulings are projected, they become the corresponding harmonic bundle.

Proof of Theorem 3. Consider a harmonic curve, $\mathcal{C}$, in a plane π. Our main aim is to prove that

- *there exists a ruled surface $\mathcal{S}$ that has $\mathcal{C}$ as a section,*

that is, such that $\mathcal{C} = \mathcal{S} \cap \pi$. In the process, we will obtain the desired polarity in π to complete the proof of the Polarity Theorem and moreover, it will yield that any such section of a ruled surface with a non-tangent plane is a harmonic curve.

By definition, $\mathcal{C}$ is generated by four points A, C, B, D in general position and paired diagonally. As before, let a and b be the generating tangents at A and B, respectively; and let $L = a \wedge b$, $\ell = A \vee B$. Observe that $D = C \cdot \rho_{L,\ell}$, so that giving a and b is equivalent to knowing D.

Choose two points P and S not in π and colinear with L (see Figure 7.a). These points are enough to construct the desired ruled surface $\mathcal{S}$; P will be the pole of π with respect to $\mathcal{S}$, and S will be a point in it.

Let S' be the harmonic fourth of S with respect to P and L; that is, $S' = S \cdot \rho_{P,\pi}$. Since $S \neq S'$, the four lines from S and S' to A and B can be colored red and blue so that only lines of opposite colors touch. And finally, consider the red (blue) line through C transversal to the two blue (red) lines. We now have a Dandelin Configuration of six lines colored red and blue: let $\mathcal{S}$ be the doubly ruled surface it defines (Figure 7.b). By construction, P and $\pi = P^\perp$ are a polar pair with respect to $\mathcal{S}$.

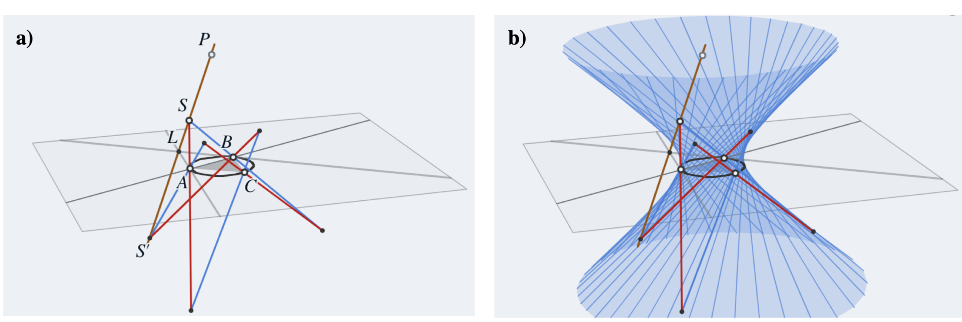

FIGURE 7. **a)** A Dandelin Configuration arising from the generators of a harmonic curve $\mathcal{C}$ in a plane π. **b)** The corresponding ruled surface that intersects π in $\mathcal{C}$.

The polarity induced by $\mathcal{S}$ restricts naturally to a polarity in the plane π as follows: the polar line of a point in π is the intersection of its polar plane with π; and the pole of a line in π is the pole of the plane it generates with P.

Since harmonic reflections preserve the planes through their center, those for non-incident polar pairs with pole in π, restrict to harmonic reflections in non-incident polar pairs of π that leave $\mathcal{S} \cap \pi$ invariant. Therefore, item (ii) of Theorem 3 follows when we prove that $\mathcal{C} = \mathcal{S} \cap \pi$.

To see this, recall the description (2) of $\mathcal{C}$. It is a parametrization with points $X \in \ell$: for $X \neq A, B$ the corresponding point in $\mathcal{C}$ can now be written $C \cdot \rho_{X,X^\perp}$. We can also view it as a parametrization of $\mathcal{C}$ with the planes $X^\perp$ about $\ell^\perp = S \vee S' = L \vee P$. Let c be one of the rules of $\mathcal{S}$ at the point C, say the red one. The point $C \cdot \rho_{X,X^\perp}$ can be obtained by taking the blue rule at $c \wedge X^\perp$ and intersecting it with π; and this description extends to $X^\perp = A^\perp, B^\perp$ to give A and B (see Figure 7). Since $c \wedge X^\perp$ pairs planes about $\ell^\perp$ with points in c, we have that points in $\mathcal{C}$ are bijectively parametrized with blue rules, via intersection with π, which is precisely $\mathcal{S} \cap \pi$. □

4. Classic theorems

We have somehow used an alternative definition of a conic curve as the section of a doubly ruled surface with a non-tangent plane. This description was Dandelin's idea to prove Pascal's Hexagon Theorem in [4], considering the alternately colored rules of the surface that an inscribed hexagon in the conic produces. The proof of Pappus Hexagon Theorem, which has the same conclusion than Pascal's, is from certain point on, literally the same; but the plane to be considered is a tangent one. We will detail this case first, for it has the extra corollary of exhibiting the equivalence of the Axiom of Double Rulings with the most classic of projective theorems.

Theorem 5 (Pappus). *Pairs of opposite sides of an hexagon whose vertices lie alternately in two coplanar lines, meet in collinear points.*

Proof. Let $A_1, B_3, A_2, B_1, A_3, B_2$ be the, cyclicly ordered, vertices of such an hexagon (see Figure 9.a). We have named them so that A_1, A_2, A_3 lie in a line, b_0 say, and the vertices B_i lie in a line a_0; let $O = A_0 = B_0 = a_0 \wedge b_0$. If we define,

$$P_i = (A_j \vee B_k) \wedge (A_k \vee B_j)\,, \tag{6}$$

for $\{i, j, k\} = \{1, 2, 3\}$, we must prove that P_1, P_2, P_3 are collinear.

Choose two auxiliary generating lines a_1, a_2 that intersect the plane $\pi = a_0 \vee b_0$ in A_1, A_2 respectively. Let us color a_0, a_1, a_2 red, as opposed to the rules in their transversal ruling, which we color blue. Let b_1, b_2, b_3 be the blue rules through B_1, B_2, B_3 ($\in a_0$), respectively. Finally, let a_3 be the red line through A_3 transversal to b_1 and b_2. By construction, all opposite colored lines meet except maybe a_3 and b_3; but these two lines meet by the Double Rulings Axiom (one could extend a_0, a_1, a_2 using b_0, b_1, b_3).

Now we have a Dandelin configuration outside of π. And from it, we get, for $\{i, j, k\} = \{1, 2, 3\}$:

$$\begin{aligned} P_i &= ((a_j \vee b_k) \wedge \pi) \wedge ((a_k \vee b_j) \wedge \pi) \\ &= ((a_j \vee b_k) \wedge (a_k \vee b_j)) \wedge \pi \\ &= ((a_j \wedge b_j) \vee (a_k \wedge b_k)) \wedge \pi\,. \end{aligned} \tag{7}$$

Therefore (see Figure 9.b), the points P_1, P_2, P_3 lie in the line

$$((a_1 \wedge b_1) \vee (a_2 \wedge b_2) \vee (a_3 \wedge b_3)) \wedge \pi\,.$$

□

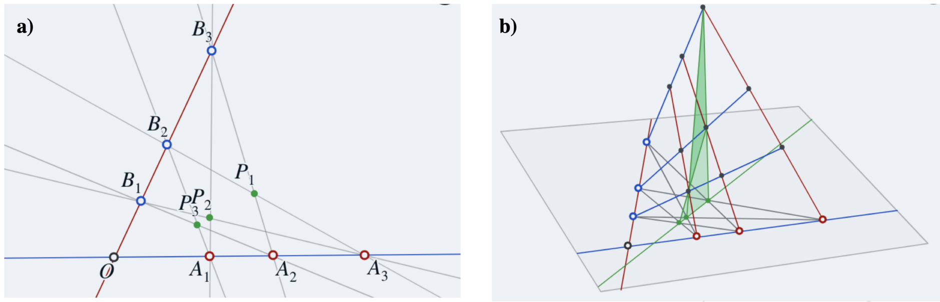

FIGURE 8. **a)** Pappus Theorem and **(b)** Dandelin's proof.

Theorem 6. *The Double Ruling Axiom is equivalent to Pappus Theorem.*

Proof. We have just proved Pappus Theorem using the Double Ruling Axiom. Now assume that Pappus Theorem holds.

The Double Ruling Axiom is clearly equivalent to the following statement: *if four lines belong to a ruling, then a rule in the transversal ruling of three of them also touches the fourth.* That is,

$$b_0, b_1, b_2, b_3 \in \mathcal{R}(a_0, a_1, a_2) \quad \text{and} \quad a_3 \in \mathcal{R}(b_0, b_1, b_2) \quad \Rightarrow \quad a_3 \text{ touches } b_3 \,.$$

So lets suppose that a_0, a_1, a_2, a_3 are mutually generating red lines and b_0, b_1, b_2, b_3 mutually generating blue lines, so that all opposite colored pairs meet except for a_3 and b_3; we must prove that they also meet.

Let $A_i = a_i \wedge b_0$ and $B_i = b_i \wedge a_0$, for $i = 1, 2, 3$. By Pappus Theorem in the plane $a_0 \vee b_0$, the three "Pappus points" (6) lie in a "Pappus line" p. The case of (7) that still holds is

$$P_3 = ((a_1 \wedge b_1) \vee (a_2 \wedge b_2)) \wedge \pi \,.$$

Consider the plane $\delta = p \vee (a_1 \wedge b_1) = p \vee (a_2 \wedge b_2)$, with lines $\ell_1 = P_1 \vee (a_1 \wedge b_1)$ and $\ell_2 = P_2 \vee (a_2 \wedge b_2)$. Finally, let $W = \ell_1 \wedge \ell_2$; it is a well defined point because both lines are in the plane δ thanks to Pappus. To see that $W \in a_3$, consider the three planes δ, $a_3 \vee b_1$ and $a_3 \vee b_2$. Using (6), one gets that their pairwise meeting lines are ℓ_1, a_3 and ℓ_2, so that a_3 passes through W. Analogously, W is the meeting point of the planes δ, $a_1 \vee b_3$ and $a_2 \vee b_3$, so that $W \in b_3$; therefore, a_3 meets b_3 and the proof is complete. □

For completeness sake, we briefly reproduce Dandelin's proof of Pascal's Theorem, [4].

Theorem 7 (Pascal)**.** *Pairs of opposite sides of an hexagon whose vertices lie in a harmonic curve, meet in collinear points.*

Proof. Let $A_1, B_3, A_2, B_1, A_3, B_2$ be the vertices of such an hexagon; the cyclic order is according to the hexagon and have nothing to do with the conic curve $\mathcal{C}$ on which they lie. Assuming the Axiom of Doubly Ruled Surfaces —or equivalently, Pappus Theorem— there exists a doubly ruled surface $\mathcal{S}$ that cuts the plane in $\mathcal{C}$. For $i = 1, 2, 3$, let a_i be the rules of one of its rulings that pass through A_i, and b_i the rules in the opposite ruling that contain B_i. The argument now follows verbatim the one we used for Pappus Theorem, once the Dandelin Configuration (without a_0 and b_0) is produced. □

A final classic theorem worth mentioning is the following. Where we should stress that the points are now completely general, and not a harmonic quadruple, and hence one more is needed.

Theorem 8. *Through five points in general position in a plane, there passes a unique harmonic curve.*

Proof. Color three of the points red and two blue. Consider two generating blue lines that cut the plane in the blue points. The three red lines incident to the red points and transversal to the chosen blue lines, generate a ruled surface that cuts the plane in a harmonic curve containing the five points.

Uniqueness has two aspects. One is combinatorial. Once there is a chosen surface, for any 3-2 coloring of the points, there is a precise choice of blue lines that yields the same surface. That it does not depend on the chosen surface follows from a construction within the plane based on Pascal's Theorem, whose details are well known. □

5. Projectivities

6. Axioms for projective geometry

References

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