HARMONY, CONICS AND PERSPECTIVE

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1. Introduction

Perspective is one of the most highly esteemed achievements of the Renaissance. The artists of that iconic period, focusing on how to draw realistic 3-D scenes on 2-D canvases, made important discoveries on how humans perceive three dimensional space, [1]. Two centuries later such knowledge was further developed by Girard Desargues to establish the foundations of *projective geometry*, a purely mathematical endeavor. Desargues was a contemporary fellow countryman of Rene Descartes, so that the dawn of projective geometry and analytic geometry are also contemporary. However, in educational terms, we mathematicians have left such cultural heritage in the realm of the arts curriculum and barely touch it in ours; Desargues, compared to Descartes, is barely known. Most mathematics students don't even know how to draw a tiled floor in perspective and how that simple ruler and pencil process is related to conics.

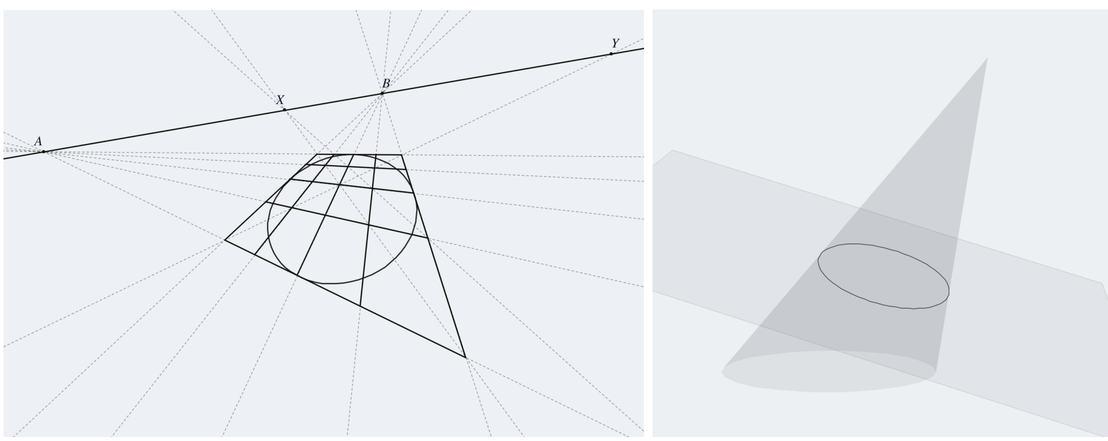

Figure 1. **a)** Tiled floor and circle in perspective. **b)** Conic section.

The standard math curriculum defines conics as intersections of planes with *circular* cones; and then studies their properties from definitions that are transformed into equations involving the cartesian (or polar) coordinates of the points in such curves, many times without showing why these two ways of looking at conics are equivalent. This approach leaves students with the wrong impression that conics are metric in nature and that the correct (and only) way to understand them is by manipulating algebraic equations.

While the algebraic equations treatment of conic sections constitutes the main content of *analytic geometry* and as such is very important and useful as a basis for Calculus, it ignores completely the purely *synthetic* geometrical aspects of conics, which are mathematically more essential and their properties constitute the main content of *projective geometry*. This is what we show in this paper.

We define conics using exclusively projective concepts: points, lines, planes and intersections. First we define projections and harmony; next we give some definitions of conics, prove their equivalence, and explore their fascinating projective properties using only synthetic (i.e. non-algebraic) methods. We also show how the metric or analytic properties of conics arise naturally when they are embedded in euclidian space. All this is achieved in the paper by adopting three fundamental ideas:

(1) Space has (at least) three dimensions.
(2) Harmony, a purely projective concept, is the central defining aspect of conics.
(3) An axiom of *double-ruling* is favored over other equivalent more traditional ones.

Let us explain briefly why these ideas are both natural and powerful.

Since the way we represent the world in drawings and paintings is by *projecting* three dimensional objects onto plane surfaces, it is not surprising that several theorems in planar *projective* geometry can be best understood and more easily proved by *3D-liftings*. A well known example is the theorem of DESARGUES stating that the corresponding sides of two coplanar triangles in perspective meet in collinear points, as shown in a) of the following figure.

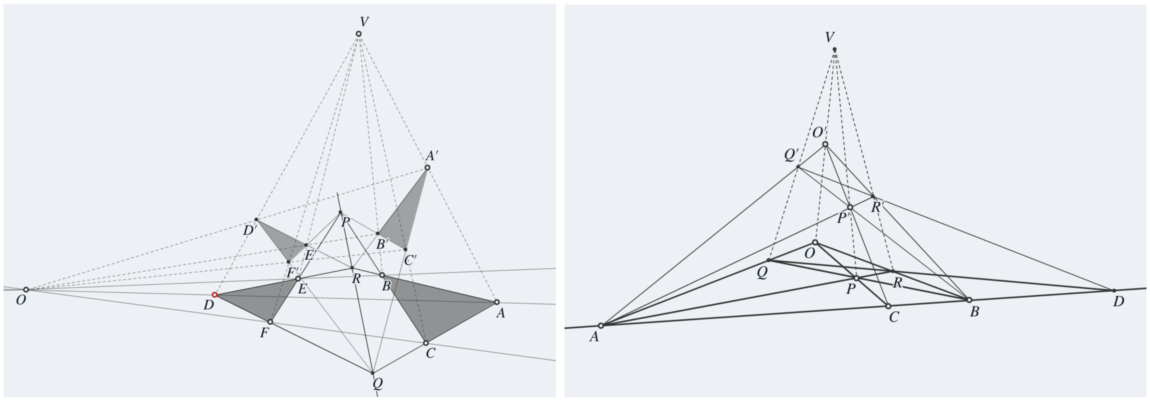

FIGURE 2. Proofs by 3D-lifting of **a)** Desargues theorem and **b)** Harmonic theorem.

It is also known that the notions of *harmonic quadruple* and *harmonic pencil* are projective in nature and need no metric or algebraic concept in order to be defined, even though they are almost always introduced through the *cross ratio*, which involves relations between measured segments. It may be not so well known, but it is a fact, that these notions may be formalized and shown to be well defined (Harmonic theorem) by using purely projective arguments. Again, such proofs are most easily achieved using *3D-lifting* procedures, as illustrated in b) of the above figure.

What may be still unknown, and we will prove in the paper, is that that conics may be defined as the locus of points that *see* four fixed points in general position (i.e. no three collinear) through a *harmonic pencil*. Since this is such a simple and general idea, we considered harmony to be the basic notion behind conics and for this reason we prefer to call them *harmonic curves* instead of conics or conic sections.

We will show that harmonic curves may also be defined as intersections of planes with ruled surfaces that are constructed from three arbitrary lines in general position (no two of them coplanar), by a purely projective (non-metric) way, using only incidence of lines and planes. The three lines determining the surface are called *generating lines*, and the surface consist of exactly those lines that touch all three *generating lines*.

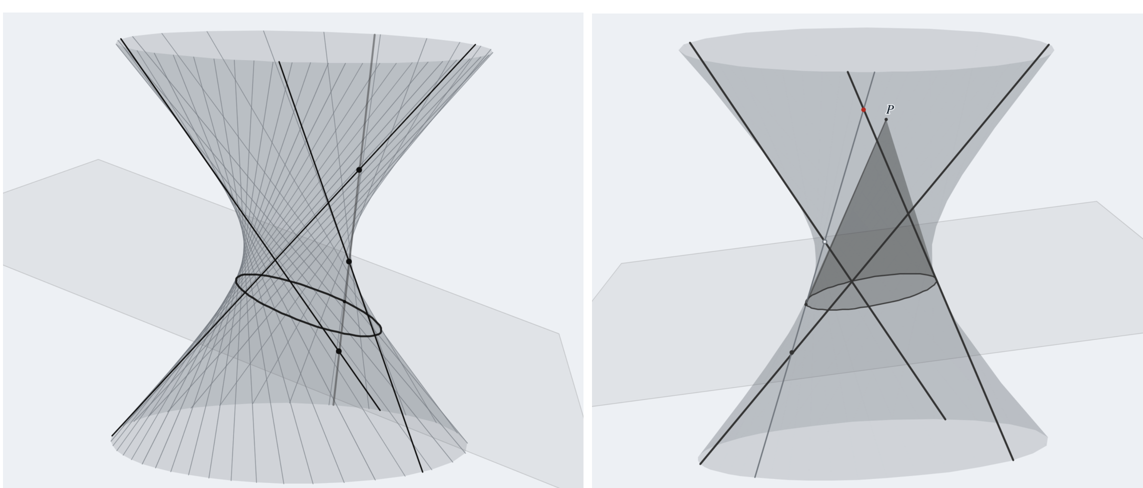

FIGURE 3. **a)** Intersection of a ruled surface with a plane. **b)** Polarity of doubly ruled surfaces.

These surfaces were introduced by HILBERT and COHN VOSSEN in their famous book *Geometry and the Imagination*, however they do not discuss the relations they have with conic sections, with Pappus' theorem and with Pascal's hexagon theorem. What we call the *axiom of double-ruling* is the statement that if any three lines touching the three generating lines of such a surface are used as generating lines for a second ruled surface, the new surface *coincides with the original one*. If this statement is assumed as an axiom and a harmonic curve is defined to be the intersection of a ruled surface with a plane, it is easy to obtain a proof of Pascal's hexagon theorem by a kind of *3D-lifting* procedure. Although this was done by GERMINAL PIERRE DE DANDELIN in the early XIX century, even today it is far from being a well known fact.

It is interesting to realize, as we will prove later on, that the axiom of double-ruling is equivalent to Pappus' theorem and thus it is as legitimate to assume it in developing projective geometry as it is to assume Pappus' theorem or any of the other equivalent statements that are commonly used for that purpose.

A third way of defining harmonic curves is through the concept of polarity, which is closely related to harmony. This was done in the XIX century by KARL GEORG CHRISTIAN VON STAUDT. In this paper we extend the concept of polarity to three dimensions, apply it on doubly-ruled surfaces, as illustrated in **b)** of the above figure, and prove that our three different definitions of harmonic curves are equivalent, obtaining in the process Von Staudt's two dimensional results on conics (i.e. harmonic curves) and polarity as corollaries.

The idea of developing projective geometry as presented in this paper was inspired by Chapter 3: The Horizon of JOHN STILLWELL'S book *Yearning for the Impossible*[7]. We wish to recognize this source of inspiration without which the present paper and many other aspects of our work, including ProGeo3D, would not exist.

2. HARMONY

Defined as a special case of the cross ratio as we still teach it today, harmonicity seems to have been first used by Apolonius related to conics (see e.g. [2]). However, it was observed to be a non metric concept deeply related to perspective until the XIX century by Karl Georg Christian von Staudt, [8]. It can be defined as follows.

Four concurrent lines form a *harmonic pencil* if there is a parallelogram with two of them as sides, a third as diagonal and the fourth parallel to the other diagonal; the concurrency point is called its *center*; O in Figure 1.a. It is a projective notion because the role played by the line at infinity, may be assumed by any line h not incident with O, called the *horizon*, to define the equivalence relation of *parallelism* as concurrency with h, as in Figure 1.b. A *harmonic range* is the intersection of a harmonic pencil with a line not incident with its center (e.g., the four points in h in Figure 1.b, divided into two pairs coming from "sides" versus "diagonals").

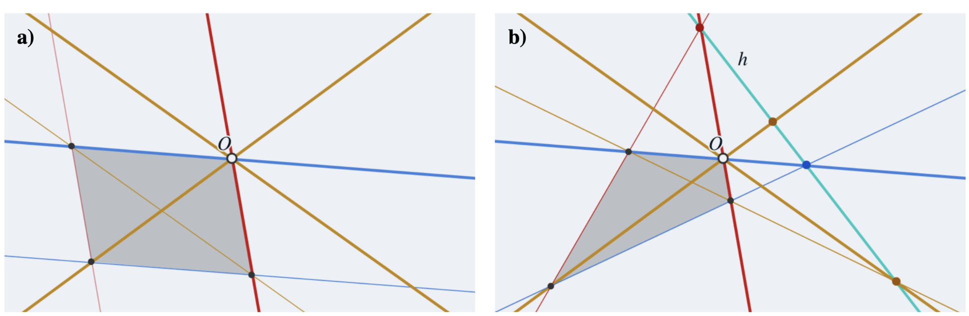

FIGURE 4. **a)** A harmonic pencil with a parallelogram construction. **b)** Parallelism may be defined by any horizon not through the concurrency point.

Dual geometric constructions that yield the *harmonic conjugate* of a point (or line) with respect to a colinear (concurrent) pair of points (lines), follow from the definitions. These constructions are also referred to as the *fourth harmonic*, because they produce a harmonic range (pencil) out of three of its elements, with one distinguished.

Harmonicity is preserved by projections. It is independent of the point of view, in the sense that all points outside the *support* (line) of a harmonic range "see" it as harmonic. However, if four points in *general position* (i.e., no three coloinear) are considered, some points "see" them as harmonic but most don't.

The *harmonic curve* generated (or defined) by four points in general position and partitioned into two pairs, is the locus of points that *see* them as harmonic, that is, which are the center of a harmonic pencil that contains them. There are *generic* points different from the generators, whose lines to them form a harmonic pencil (Figure 2.a); and the generators which "see themselves" in the fourth harmonic of its lines to the other three (the one in their pair, distinguished); these lines are called the *generating tangents* and are paired by incidence with the generating points (Figure 2.b).

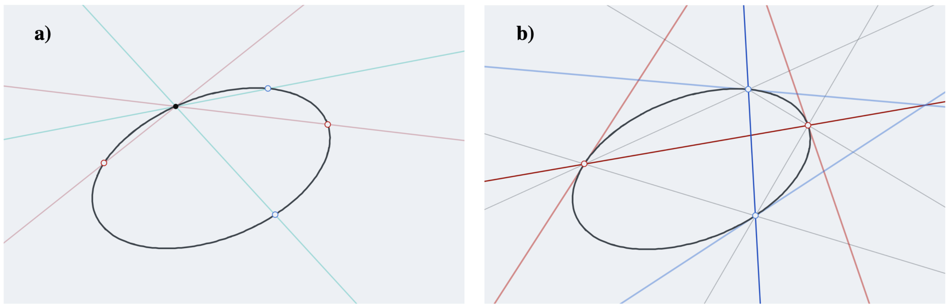

FIGURE 5. **a)** The harmonic curve generated by four points, two red and two blue; **b)** they belong to the harmonic curve because of their tangent lines.

A dual notion arises naturally. The *harmonic bundle* generated (or defined) by four lines in general position (no three collinear) and partitioned into two pairs, is the set of lines that *see* (or *feel*) them as harmonic, that is, that contain a harmonic range *transversal* (i.e., paired by incidence) with the lines. Again, the *generic* lines intersect the generators in a harmonic range, and the generating lines "choose" the fourth harmonic of their intersection with the other three, to "represent" them; it is their *contact point*.

Of course, it is true that each line in the harmonic bundle that arises from the generating tangents of a harmonic curve, passes through a unique point in it (its *contact point* and the line is the *tangent* there). But the proof of this (Theorem 2) needs more work, so we shall address our main issue first.

Theorem 1. *Conics are harmonic curves.*

Proof. First observe that the vertices of a square paired diagonally, generate its circumscribed circle as harmonic curve; it is a high-school exercise (see Figure 3.a). Then, because harmonicity is preserved by projections, the plane section of any cone with a circle as base, is the harmonic curve of the projected generators of the circle (see Figure 3.b). And conics are, by their classic definition, such sections for perpendicular, or straight, cones. □

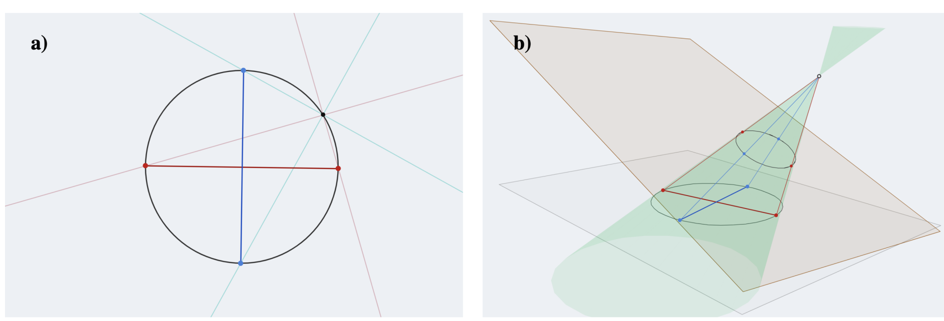

FIGURE 6. **a)** A circle is the harmonic curve generated by the extremes of two perpendicular diameters. **b)** The projection of a harmonic curve is also a harmonic curve; hence, conics are harmonic curves.

A gap in this proof is that it assumes that no other point outside the circle belongs to the harmonic curve. This will be taken care of with the following construction of a harmonic curve.

Given four points in general position and partitioned into the pairs A, B and C, D, let $\mathcal{C}$ denote the harmonic curve they generate, and let ℓ be the line through A and B (see Figure 4.a). Given a point $X \in \ell$, let Y be its fourth harmonic with respect to A, B. Let Z be the intersection of the lines XC and YD. A more convenient notation, because it makes duality easier to see, and that we will adopt is

$$Z = (C \vee X) \wedge (D \vee Y)\,; \tag{1}$$

where $\vee$, "join", denotes the linear generating (or closure) operator and $\wedge$, "meet", is intersection. Then, Z sees the generators as harmonic ($Z \in \mathcal{C}$), because its lines to them intersect ℓ in the harmonic range A, X, B, Y (observe that it is written so that the pairs are not consecutive, which is the cyclic order with which they appear on ℓ).

So that *as X moves along ℓ, Z traces all of $\mathcal{C}$*; because in the line $X \vee C$ there is no point in $\mathcal{C}$, other than Z or C. This implies that $\mathcal{C}$ is bijectively parametrized by the projective line ℓ; or equivalently, by the concurrent pencil of lines centered at C. Therefore, the circumscribed circle is indeed the harmonic curve of the vertices of a square, paired diagonally.

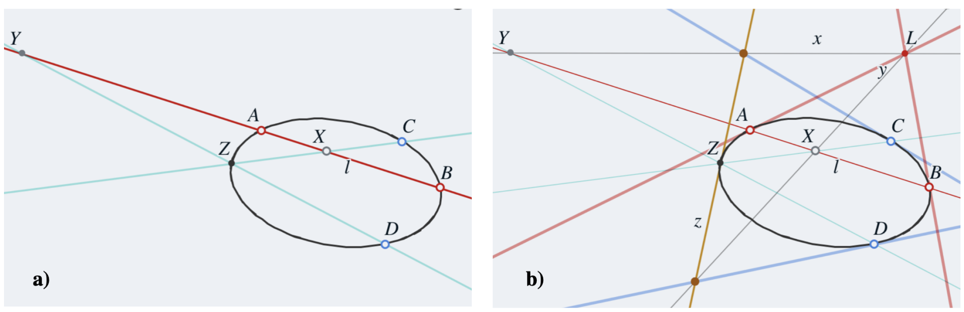

FIGURE 7. **a)** The construction of a harmonic curve. **b)** The dual construction for its harmonic bundle. The point Z and the tangent line through it, z, are the image of D and its tangent line, d, under the harmonic reflection with mirror $y = X \vee L$ and center Y.

With a dynamic geometry system having the fourth harmonic built in as a tool (such as *ProGeo3D*, [6], used to produce the figures), this construction is quite simple. It also derives into insightful examples of parabolas and hyperbolas by sending some of the starting elements of the construction to infinity.

We have proved that circles drawn in perspective are the classic conic curves. Now, the three types fall into where the observer may be. If she is outside the circle, she sees an ellipse; from the inside, he sees a hyperbola, and the parabola (having the line at infinity as a tangent) is the unstable passage from one case to the other.

To see that harmonic curves are paired naturally with harmonic bundles, we need yet another important basic concept associated with harmony. Given a line m and a point $C \notin m$, the *harmonic reflection* with *mirror* m and *center* C, denoted $\rho_{C,m}$, is the map from the projective plane to itself that sends a point to its fourth harmonic with respect to C and the intersection with the mirror m of its line to C.

In other words, consider the fourth harmonic with respect to a pair of points (or with that "mirror pair"), as a map of a projective line to itself, then glue these maps on the pencil of lines about the center C with the other mirror point at the mirror m which thus remains pointwise fixed; it looks as a reflection close to the mirror and as a central inversion about the center. Of course, harmonic reflections are *colinearities* (send lines to lines), their duals are also harmonic reflections (with roles interchanged) and are *involutions* (they are their own inverse). Furthermore, they can be naturally generalized to 3D with a plane as mirror and a non-incident point as center; and they have as particular cases, classic euclidian reflections and central inversions.

We need a basic fact about harmonic reflections which we call Klein's triangle lemma, because it associates a Klein group (of four elements) to a triangle. It follows directly from the fact that projections preserve harmonic ranges; we leave its proof as an exercise.

Lemma 1 (Klein's triangle)**.** *Given a triangle, the composition of the three harmonic reflections with one vertex as center and the opposite side as mirror is the identity.*

Let us now address the pairing between harmonic curves and harmonic bundles.

Theorem 2 (Duality curves-bundles)**.** *Points in a harmonic curve are* paired *(in bijective correspondence) by incidence with the lines in the harmonic bundle defined by its generating tangents.*

Proof. If $\mathcal{C}$ is the harmonic curve generated by the points A, C, B, D, and a, c, b, d are their respective generating tangents, we will prove that the pairing by incidence between the defining points and lines (expressed as upper and lower case), extends to all the points in the curve and the lines in the corresponding harmonic bundle $\mathcal{C}^*$.

In Figure 4.b, the dual construction to obtain the bundle is depicted. By Klein's triangle lemma (Lemma 1) on the triangle LXY (where $L = a \wedge b$), with respective opposite sides ℓxy, we have that

$$C \cdot \rho_{X,x} = C \cdot (\rho_{L,\ell} \cdot \rho_{Y,y}) = (C \cdot \rho_{L,\ell}) \cdot \rho_{Y,y} = D \cdot \rho_{Y,y}\,,$$

where we act and compose on the right, and denote both with "$\cdot$".

Therefore, for $X \neq A, B$, we can also express Z as $C \cdot \rho_{X,x} = D \cdot \rho_{Y,y}$ because Z was defined in (1) as the intersection of the lines from the center of the harmonic reflection to the point acted upon, on both sides of the equation. The dual expressions for the line

$$z = c \cdot \rho_{X,x} = d \cdot \rho_{Y,y} = (c \wedge x) \vee (d \wedge y)\,,$$

yield that for each generic point, $Z = C \cdot \rho_{X,x}$, in the harmonic curve $\mathcal{C}$, there is an incident line, $z = c \cdot \rho_{X,x}$, called its *tangent*, in the harmonic bundle $\mathcal{C}^*$ because they are the image of the incident pair $C \in c$ under the same harmonic reflection $\rho_{X,x}$. □

As a corollary to the proof we must remark that, with the notation of Figure 4.b used in the preceding theorem, the harmonic curve can also be expressed as

$$\mathcal{C} = \{\, C \cdot \rho_{X,x} \,|\, X \in \ell \setminus \{A, B\}\,\} \cup \{A, B\}\,, \tag{2}$$

where the points A and B must be excluded as parameters of the first set, in order to have the harmonic reflection $\rho_{X,x}$ well defined; but then, they must be added back again to obtain $\mathcal{C}$. Recall that the line x was defined as $x = L \vee Y$, where $L = a \wedge b$ and Y is the fourth harmonic of $X \in \ell = A \vee B$ with respect to A, B; so that all the harmonic reflections, $\rho_{X,x}$, used in (2), interchange the points A and B as well as their generating tangents a and b. They send C to a point in the harmonic curve $\mathcal{C}$. But moreover, we will see that they are symmetries of all of $\mathcal{C}$. The pairing $X \leftrightarrow x$ of points in ℓ and lines through L, is yet another example, other than Theorem 2, of the "polarity" that a harmonic curve induces on the plane. To sate this properly, we need the definition:

A *polarity* is a pairing of points and lines (the terms *pole* and *polar* are used) that preserves incidence (or reverses inclusion)[1].

Theorem 3 (Polarity)**.** *A harmonic curve $\mathcal{C}$ induces a polarity (expressed by upper and lower case of the same letter) satisfying:*

i) $P \in \mathcal{C} \Leftrightarrow P \in p\,.$
ii) *If $P \notin \mathcal{C}$ then the harmonic reflection $\rho_{P,p}$, with P as center and its non-incident polar line p as mirror, leaves $\mathcal{C}$ invariant.*

We have already seen item (i) as Theorem 2; tangent lines to a harmonic curve are defined as their polars. The rest of the proof will be given towards the end of the next section; for the moment, let us make three pertinent remarks about the theorem itself.

First, two mathematicians directly associated to this theorem are Jean-Victor Poncelet and Karl G. C. von Staudt. The first, proved the relation of poles and polars of conic sections with harmony, and the second, soon after, developed polarities as a general concept and used it as an alternative way to define conic curves within projective geometry and with no relation to metric or algebraic considerations. This definition is the one Coxeter uses in his influential book [3], and calls it "most appealing" because it has duality built into it. Two types of polarities must be precised: *euclidian* when no point is incident with its polar line, and *hyperbolic* when there exist pole and polar incident pairs; the terms are then related to the groups generated

[1] An extra hypothesis is used in [3]; namely that for some line, the pairing of its point range to the line pencil of its pole be a projectivity. But we do not need to stress this issue.

by harmonic reflections on non-incident polar pairs. Von Staudt's definition is equivalent to item (i) of the theorem in a hyperbolic polarity, while Poncelet's results can be rephrased as item (ii).

Second, as examples of polar pairs, we have named lines and points in Figure 4.b according to the upper and lower case rule for poles and polars with respect to the displayed harmonic curve.

Third, in terms of the polarity induced by a harmonic curve, the point quadruples that generate it as such, are those for which the pole of the line of one pair is incident with the other line (in Figure 4.b: $(c \wedge d) \in (A \vee B) = \ell$ and thus, $L = (a \wedge b) \in (C \vee D)$, see also Figure 2.b). They could rightfully be called *harmonic quartets* in $\mathcal{C}$; and in the Klein-Beltrami model of the hyperbolic plane, they correspond to extremes or "points at infinity" of perpendicular lines. In the circle, they are called "cyclic quadrangles" (?).

3. Ruled surfaces

Our approach to prove the polarity theorem (3) is inspired by Germinal Pierre Dandelin's proof of Pascal's hexagonal theorem, [4]. He considers the plane within three dimensional space and then makes his arguments there. Of course, this general idea is classic. It is typified by Desargues' theorem which becomes almost obvious in three dimensions, and it is quite useful to prove other basic results such as the projective invariance of harmony. It is a natural idea because perspective, as a basic source of projective geometry, makes little sense without three dimensions. Also, conic sections use three dimensional space to be defined in the classic way, but in Dandelin's proof, instead of circular cones, ruled surfaces are used.

We first review Hilbert and Cohn-Vossen's construction of ruled surfaces in [5], which appeared in print almost a century after Dandelin's proof, and made clear that they can be defined depending only on incidence.

Consider two lines, say a and b, in three dimensional projective space. They touch if and only if they are coplanar. If this is not the case, they can be called a *generating* pair because for any point X not in them, there is a unique line through X *transversal* (i.e., with a common point) to a and b; namely,

$$(X \vee a) \wedge (X \vee b)\,,$$

where we are extending the use of the operations $\vee$ and $\wedge$ to all flats, which is their natural environment.

Now consider three lines a, b, c in *general position* (i.e., each pair is generating). The *transversal ruling* to a, b, c is the set of lines that are transversal to them (i.e., that touch the three); any such set of lines will be called a *ruling* (see Figure 5.a) and its elements are called *rules*. If we denote it $\mathcal{R} = \mathcal{R}(a, b, c)$, the above observation implies that $\mathcal{R}$ is parametrized by incidence with the points in any of the three generating lines (through any point in them there pases a unique rule). It will be important to note that, dually, $\mathcal{R}$ is also parametrized by planes containing one of the lines; if we denote planes by greek letters (points and lines are, respectively, upper and lower case latin) we have, for example, that

$$\mathcal{R}(a, b, c) = \{\, (b \wedge \alpha) \vee (c \wedge \alpha) \,|\, a \subset \alpha \,\}\,. \tag{3}$$

Every pair of rules in $\mathcal{R}$ is generating; otherwise, their three transversal lines a, b, c would be coplanar. Thus, for any triplet $a', b', c' \in \mathcal{R}$ we get a transversal rulling $\mathcal{R}' = \mathcal{R}(a', b', c')$ that contains the original three lines, a, b, c; this ruling is an *extension* of a, b, c (see Figure 5.b). In real projective space (the one outlined by Desargues according to our spatial experience and intuition, by incorporating a plane at infinity to euclidian space), it is true that there is only one extension to a ruling of three lines in general position. But there is no simple, or elementary, proof of this fact. Therefore, we state it as an axiom that will be proved to be equivalent to Pappus Theorem and other classic statements that have been used as axioms.

Axiom of double rulings.[2] *Three lines in general position belong to a unique ruling.*

[2]In Spanish, we call this axiom *"Axioma del Equipal"*, refering to a classic, mexican style of furniture that uses double rulings for bases.

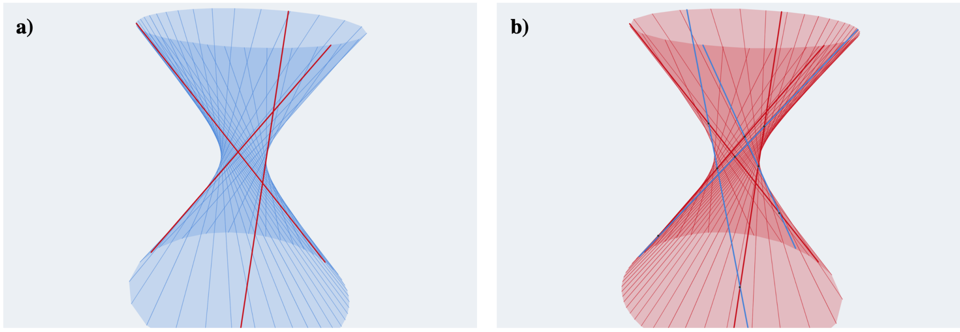

FIGURE 8. **a)** The transversal ruling (in blue) to three red lines. **b)** The transversal ruling to any three blue rules (in red) contains the three red lines.

The name arises because this axiom immediately implies that rulings are paired: any ruling comes with an *opposite ruling* (the transversal ruling to any triplet of its rules). The *doubly ruled surface* (we also refer to it simply as a *ruled surface*) obtained as the union of the rules in a ruling is also the union of the rules in its opposite ruling. They are usually called hyperboloids of one sheet, but also the hyperbolic paraboloids are a special type (they are tangent to the plane at infinity).

Every point on a ruled surface has a *tangent plane* associated to it: the one generated by the unique rules through the point in the two rulings of the surface. If we consider it as its *polar plane*, this association gives rise to a full scale *polarity*: a pairing between points and planes that preserves incidence. That is, we now prove a three dimensional analogue of the Polarity Theorem (3):

Theorem 4 (Polarity of ruled surfaces)**.** *The pairing of points in a ruled surface $\mathcal{S}$ with their respective tangent planes, extends to a polarity of projective space, in which if $P \notin \mathcal{S}$ then P is not incident with its polar plane π, and the harmonic reflection $\rho_{P,\pi}$, with P as center and π as mirror, leaves $\mathcal{S}$ invariant.*

Proof. Consider a specific ruling $\mathcal{R}$. It has an opposite ruling $\mathcal{R}'$ and the doubly ruled surface they define is

$$\mathcal{S} = \bigcup_{x\in\mathcal{R}} x = \bigcup_{y\in\mathcal{R}'} y\,.$$

The basic observation that guides the proof is that any point P not in $\mathcal{S}$ and, dually, that any non-tangent plane π, induce natural abstract bijections (or *matchings*) between the two rulings of $\mathcal{S}$ by incidence:

$$\begin{array}{c} \mathcal{R} \xleftrightarrow{(P)} \mathcal{R}' \\ x \leftrightarrow y \\ \Updownarrow \\ P \in x \vee y \end{array} \quad , \quad \begin{array}{c} \mathcal{R} \xleftrightarrow{(\pi)} \mathcal{R}' \\ x \leftrightarrow y \\ \Updownarrow \\ x \wedge y \in \pi \end{array} \quad . \tag{4}$$

So that the polarity induced by $\mathcal{S}$ that we seek is the one in which P and π are a polar pair if and only if their corresponding abstract matchings by incidence (4) between opposite rulings are identical. And moreover, we have to prove that this matching is also induced by a global geometric map: the harmonic reflection $\rho_{P,\pi}$, with P as center and π as mirror.

To define the polarity induced by $\mathcal{S}$ in its complement, consider a point $P \notin \mathcal{S}$ —we could start, dually, with a non-tangent plane. Fix three rules a, b, c in the ruling $\mathcal{R}$, and beware that we have inverted the notational use of primes: their transversal ruling is now $\mathcal{R}'$.

Let $\alpha = a \vee P$. In view of (3), there is a well defined rule $a' \in \mathcal{R}'$ for which $P \in a \vee a' = \alpha$ (namely, $a' = (b \wedge \alpha) \vee (c \wedge \alpha)$ which is the line that corresponds to a under the matching (P) in (4)); let $A = a \wedge a'$. Analougously, we obtain $b', c' \in \mathcal{R}'$, for which $P \in b \vee b' = \beta$ and $P \in c \vee c' = \gamma$; let $B = b \wedge b'$, $C = c \wedge c'$. To have the same matchings (4), the polar plane to P has to be defined as

$$\pi = A \vee B \vee C\,.$$

Observe that if we had started, dually, with this plane π we would have found P as the intersection of the three tangent planes.

Now, we will show that the harmonic reflection with P as center and π as mirror, $\rho_{P,\pi}$, interchanges the lines a, b, c with the corresponding a', b', c' in the opposite ruling. By the triangular symmetry of the construction, it will be enough to prove that:

- *in the tangent plane to A, $\alpha = a \vee a'$, the pair of lines a and a' are harmonic to $A \vee P$ and $\alpha \wedge \pi$;*

because this happens if and only if $\rho_{P,\pi}$ interchanges the lines a and a'.

We have distinguished, within the general setting of a doubly ruled surface, what we will call a *Dandelin configuration*: six lines of two *types* or *colors*, three of each —red and blue in the pictures, or unprimed and primed in the text— such that a pair of them touch if and only if they have opposite types. This produces nine *basic* points and nine *tangent* planes by the "wedge" ($\wedge$) or "join" ($\vee$) of lines of different colors; but it also comes with a derived configuration of other lines and planes that naturally arise from them. That geometric-combinatorial richness is what Dandelin exploited in [4]; and we follow suit.

The tangent plane $\alpha = a \vee a'$ contains five of the nine basic points of our Dandelin configuration. The *α-quadrangle*:

$$a \wedge b', b \wedge a', a \wedge c', c \wedge a',$$

with its center $A = a \wedge a'$ where its diagonals a and a' meet. We shall see that the remaining four basic points outside of α, group naturally into two pairs whose generated lines are concurrent with the two pairs of opposite sides of the α-quadrangle.

One pair is $b \wedge c'$ and $c \wedge b'$, whose line $((b \wedge c') \vee (c \wedge b'))$ passes through P because it is precisely $\beta \wedge \gamma$. To see this, observe that both points lie on both planes (this argument will be repeatedly used), and by construction, we have $\alpha \wedge \beta \wedge \gamma = P$ (see Figure 6.a). Therefore, within α:

$$P = (\alpha \wedge \beta) \wedge (\alpha \wedge \gamma) = ((a \wedge b') \vee (b \wedge a')) \wedge ((a \wedge c') \vee (c \wedge a')).$$

The remaining pair of points are $b \wedge b' = B$ and $c \wedge c' = C$. They are both on the tangent planes $b \vee c'$ and $c \vee b'$ (see Figure 6.b). Therefore, these two planes meet α in $Q = (B \vee C) \wedge \alpha$; which can also be expressed as the intersection of their intersecting lines with α:

$$Q = (\alpha \wedge (b \vee c')) \wedge (\alpha \wedge (c \vee b')) = ((b \wedge a') \vee (a \wedge c')) \wedge ((c \wedge a') \vee (a \wedge b')).$$

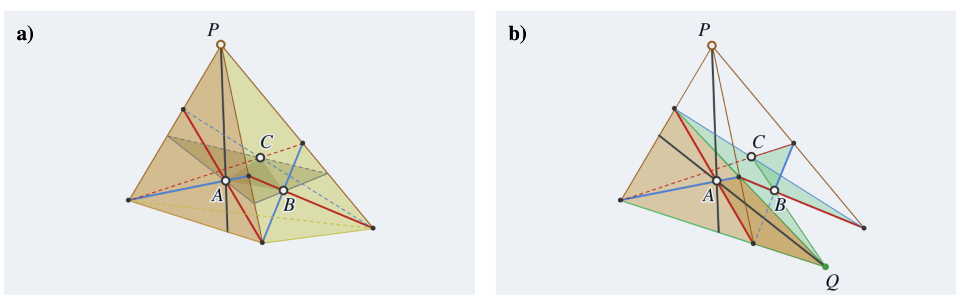

FIGURE 9. **a)** A Dandelin Configuration with a fixed matching of the two types of rules, given by the point P and the plane $\pi = A \vee B \vee C$. **b)** The harmonic pencil $a, A \vee P, a', \alpha \wedge \pi = A \vee Q$, centered at $A = a \wedge a'$ in the plane $\alpha = a \vee a'$.

The configuration in α that we have described, consisting of 7 points and 8 lines, proves (according to Figure 1.b) what we wanted: a and a' are harmonic with respect to $A \vee P$ and $A \vee Q = \alpha \wedge \pi$.

Thus, $\rho_{P,\pi}$ interchanges the rules a and a'. Analogously, it interchanges b, c with b', c' respectively. Then, it gives a bijection between the transversal rulings of a, b, c and a', b', c', which are $\mathcal{R}'$ and $\mathcal{R}$ respectively, because a line transversal to a, b, c is sent by $\rho_{P,\pi}$ to a line transversal to a', b', c' and viceversa. In particular, since a harmonic reflection sends a line to a line concurrent with the mirror and coplanar with the center, this geometric bijection corresponds to the abstract matchings (4); thus, the polar plane π to the point $P \notin \mathcal{S}$ does not depend on the choice of generating rules a, b, c.

Therefore, $\rho_{P,\pi}$ leaves $\mathcal{S}$ invariant, as we wished to prove.

Finally, we have to prove that the constructed pairing between points and planes preserves incidence. Let us denote the now well defined *polar plane* of any point X by $X^\perp$.

Let P and Q be two points such that $Q \in P^\perp$, we are left to prove that $P \in Q^\perp$.

It has to be argued in cases. The first one is when both P and Q are in $\mathcal{S}$, but then $Q \in P^\perp$ iff they belong to the same rule, because $\mathcal{S} \cap P^\perp$ is the union of the two rules at P. The case when one point is in $\mathcal{S}$ and the other is not, has been dealt with. For we have just used that given $P \notin \mathcal{S}$, the criterion which determines if Q is in $\mathcal{S} \cap P^\perp$ is precisely $P \in Q^\perp$. So, we are left to prove the generic case in which P and Q are not in $\mathcal{S}$.

It is not hard to argue that within the plane $P^\perp$, there exists a line ℓ that passes through Q and contains two different points A, B in $\mathcal{S}$. Let a, a' (and b, b') be the two rules of $\mathcal{S}$ that pass through A (B). Then, $A \in P^\perp$ and $B \in P^\perp$ imply

$$P \in A^\perp \wedge B^\perp = (a \vee a') \wedge (b \vee b') = (a \wedge b') \vee (b \wedge a') \,. \tag{5}$$

Both points in the last expression are in $\mathcal{S}$. Furthermore, we have that $(a \wedge b') \in Q^\perp$ because $Q \in \ell = A \vee B \subset (a \vee b') = (a \wedge b')^\perp$; and likewise $(b \wedge a') \in Q^\perp$. Therefore, (5) implies $P \in Q^\perp$. □

Observe that, because of the incidence invariance, the polarity extends naturally to a pairing of lines. The polar of a line ℓ is the intersection $\ell^\perp$ of all the polar planes of its points, or of any two of them.

This polarity theorem asserts that what one sees as the contour of a ruled surface is exactly its section with the polar plane of the viewpoint. Sections and projections match. We now prove that they are harmonic curves, and the corresponding harmonic bundles are the projection of the rulings.

Proof of Theorem 3. Consider a harmonic curve, $\mathcal{C}$, in a plane π. Our basic aim is to prove that

- *there exists a ruled surface $\mathcal{S}$ that has $\mathcal{C}$ as a section,*

that is, such that $\mathcal{C} = \mathcal{S} \cap \pi$. In the process, we will obtain the desired polarity in π to complete the proof of the Polarity Theorem and moreover, it will yield that any such section of a ruled surface with a non-tangent plane is a harmonic curve.

By definition, $\mathcal{C}$ is generated by four points A, C, B, D in general position and paired diagonally. As before, let a and b be the generating tangents at A and B, respectively; and let $L = a \wedge b$, $\ell = A \vee B$. Observe that $D = C \cdot \rho_{L,\ell}$, so that giving a and b through A and B respectively, is equivalent to knowing D.

Choose two points P and S not in π and colinear with L (see Figure 7.a). These points are enough to construct the desired ruled surface $\mathcal{S}$; P will be the pole of π with respect to $\mathcal{S}$, and S will be a point in it.

Let S' be the fourth harmonic of S with respect to P and L; that is, $S' = S \cdot \rho_{P,\pi}$. Since $S \neq S'$, the four lines from S and S' to A and B can be colored red and blue so that only lines of opposite colors touch. And finally, consider the red (blue) line through C transversal to the two blue (red) lines. We now have a Dandelin configuration of six lines colored red and blue: let $\mathcal{S}$ be the doubly ruled surface it defines (Figure 7.b). By construction, P and $\pi = P^\perp$ are a polar pair with respect to $\mathcal{S}$.

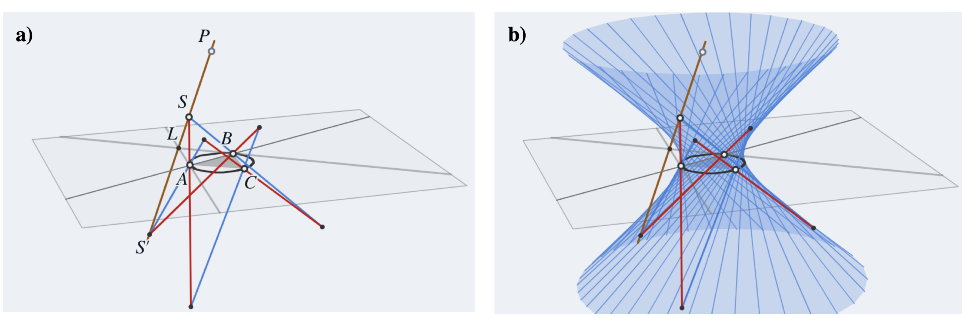

FIGURE 10. **a)** A Dandelin Configuration arising from the generators of a harmonic curve $\mathcal{C}$ in a plane π. **b)** The corresponding ruled surface that intersects π in $\mathcal{C}$.

The polarity induced by $\mathcal{S}$ restricts naturally to a polarity in the plane π as follows: the polar line of a point in π is the intersection of its polar plane with π; and the pole of a line in π is the pole of the plane it generates with P —or the intersection with π of its polar line.

Since harmonic reflections preserve the planes through their center, those for non-incident polar pairs with pole in π, restrict to harmonic reflections of π that leave $\mathcal{S} \cap \pi$ invariant. Therefore, item (ii) of Theorem 3 follows when we prove that $\mathcal{C} = \mathcal{S} \cap \pi$.

To see this, recall the description (2) of $\mathcal{C}$. It is a parametrization with points $X \in \ell = A \vee B$: for $X \neq A, B$ the corresponding point in $\mathcal{C}$ can now be written $C \cdot \rho_{X,X^\perp}$. We can also view it as a parametrization of $\mathcal{C}$ with the planes $X^\perp$ about $\ell^\perp = S \vee S' = L \vee P$. Let c be one of the rules of $\mathcal{S}$ at the point C, say the red one. The point $C \cdot \rho_{X,X^\perp}$ can then be obtained by taking the blue rule at $c \wedge X^\perp$ and intersecting it with π; and this description extends to $X^\perp = A^\perp, B^\perp$ to give A and B (see Figure 7). Since $X^\perp \mapsto c \wedge X^\perp$ pairs planes about $\ell^\perp$ with points in c, we conclude that points in $\mathcal{C}$ are bijectively parametrized with blue rules via intersection with π, which is precisely $\mathcal{S} \cap \pi$. □

Observe that, within the above framework, for any point in $\mathcal{S} \cap \pi$ the intersection with π of its tangent plane to $\mathcal{S}$ is the projection to π from P of any of its two rules. So that we may state the following theorem as a corollary to the preceding proofs.

Theorem 5. *Harmonic curves are the sections of ruled surfaces with non-tangent planes. Moreover, harmonic bundles are the projection of rulings, and the dual harmonic bundle of a section of a ruled surface is the projection from the corresponding pole of any of its two rulings.* □

4. Classic theorems

We have seen that an alternative definition of a conic section is as the intersection of a doubly ruled surface with a non-tangent plane. This description was Dandelin's idea to prove Pascal's hexagon theorem in [4], and then to consider the alternately colored rules of the surface that an inscribed hexagon in the conic produces; which we have called a Dandelin configuration. The proof of Pappus hexagon theorem, which has the same conclusion than Pascal's, is from a certain point on, literally the same; but the plane to be considered is a tangent one. We prefer to detail this latter case, for it will expose as a corollary, the equivalence of the axiom of double rulings with the most classic of projective theorems.

Theorem 6 (Pappus)**.** *Pairs of opposite sides of a hexagon whose vertices lie alternatively in two coplanar lines, meet in collinear points.*

Proof. Let $A_1, B_3, A_2, B_1, A_3, B_2$ be the, cyclicly ordered, vertices of such a hexagon (see Figure 9.a). We have named them so that A_1, A_2, A_3 lie in a line, b_0 say, and the vertices B_i lie in a line a_0; let $O = A_0 = B_0 = a_0 \wedge b_0$. If we define for $\{i, j, k\} = \{1, 2, 3\}$ the *Pappus points* of the hexagon to be:

$$P_i = (A_j \vee B_k) \wedge (A_k \vee B_j)\,; \tag{6}$$

we must prove that P_1, P_2, P_3 are collinear.

Going out to a third dimension, choose two auxiliary generating lines a_1, a_2 that intersect the plane $\pi = a_0 \vee b_0$ in A_1, A_2 respectively. Let us color a_0, a_1, a_2 red, as opposed to the rules in their transversal ruling, which we color blue. Let b_1, b_2, b_3 be the blue rules through B_1, B_2, B_3 ($\in a_0$), respectively. Finally, let a_3 be the red line through $A_3 \in b_0$ transversal to b_1 and b_2. By construction, all opposite colored lines meet except maybe a_3 and b_3; but these two lines meet by the double rulings axiom (one could extend a_0, a_1, a_2 to a ruling using, b_0, b_1, b_3 or b_0, b_2, b_3 instead of b_0, b_1, b_2 as we did).

Now we have a Dandelin configuration outside of π. And from it, and (6), we get for $\{i, j, k\} = \{1, 2, 3\}$:

$$\begin{aligned} P_i &= ((a_j \vee b_k) \wedge \pi) \wedge ((a_k \vee b_j) \wedge \pi) \\ &= ((a_j \vee b_k) \wedge (a_k \vee b_j)) \wedge \pi \\ &= ((a_j \wedge b_j) \vee (a_k \wedge b_k)) \wedge \pi\,. \end{aligned} \tag{7}$$

Therefore (see Figure 9.b), the Pappus points P_1, P_2, P_3 lie in the line

$$((a_1 \wedge b_1) \vee (a_2 \wedge b_2) \vee (a_3 \wedge b_3)) \wedge \pi\,.$$

□

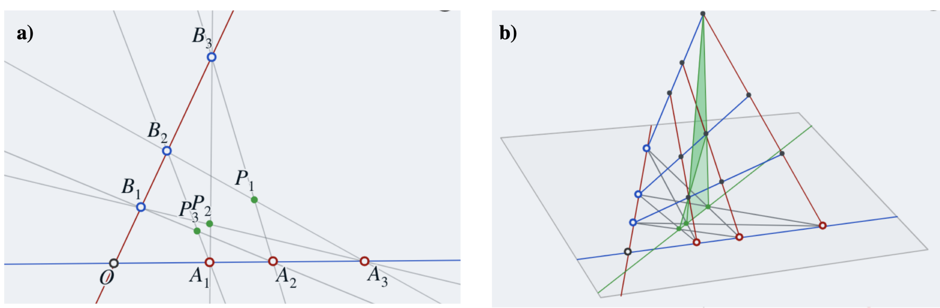

FIGURE 11. **a)** Pappus theorem and **(b)** Dandelin's proof.

Theorem 7. *The double ruling axiom is equivalent to Pappus theorem.*

Proof. We have just proved Pappus theorem using the double ruling axiom. Now assume that Pappus theorem holds.

The double ruling axiom is clearly equivalent to the following statement: *if four lines belong to a ruling, then a rule in the transversal ruling of three of them also touches the fourth.* That is,

$$b_0, b_1, b_2, b_3 \in \mathcal{R}(a_0, a_1, a_2) \quad \text{and} \quad a_3 \in \mathcal{R}(b_0, b_1, b_2) \quad \Rightarrow \quad a_3 \text{ touches } b_3 \,.$$

So lets suppose that a_0, a_1, a_2, a_3 are mutually generating red lines and b_0, b_1, b_2, b_3 mutually generating blue lines, and that we know that all opposite colored pairs meet except for a_3 and b_3; we must prove that they also meet.

Let $A_i = a_i \wedge b_0$ and $B_i = b_i \wedge a_0$, for $i = 1, 2, 3$. By Pappus theorem in the plane $a_0 \vee b_0$, the three "Pappus points" (6) lie in the "Pappus line" p. The case of (7) that still holds is

$$P_3 = ((a_1 \wedge b_1) \vee (a_2 \wedge b_2)) \wedge \pi \,.$$

Consider the plane $\delta = p \vee (a_1 \wedge b_1) = p \vee (a_2 \wedge b_2)$, with lines $\ell_1 = P_1 \vee (a_2 \wedge b_2)$ and $\ell_2 = P_2 \vee (a_1 \wedge b_1)$ (in Figure 8.b, δ is the green plane).

Let $W = \ell_1 \wedge \ell_2$; it is a well defined point because both lines are in the plane δ due to Pappus theorem. To see that $W \in a_3$, consider the planes $a_3 \vee b_1$, $a_3 \vee b_2$ and δ. Using (6), one gets that their pairwise intersection lines are a_3, $\ell_1 = (a_3 \vee b_2) \wedge \delta$ and $\ell_2 = \delta \wedge (a_3 \vee b_1)$. Since these three planes meet in a single point, we get that a_3 passes through W. Analogously, $W \in b_3$ because W is the meeting point of the planes $a_1 \vee b_3$, $a_2 \vee b_3$ and δ. Therefore, a_3 meets b_3 in W. □

For completeness sake, we reproduce Dandelin's original proof of Pascal's theorem, [4]; which, by the way, seems to be the first traceable one in the literature.

Theorem 8 (Pascal)**.** *Pairs of opposite sides of a hexagon whose vertices lie in a harmonic curve, meet in collinear points.*

Proof. Let $A_1, B_3, A_2, B_1, A_3, B_2$ be the vertices of such an hexagon; the cyclic order is according to the hexagon and have nothing to do with the conic curve $\mathcal{C}$ on which they lie. By Theorem 5, there exists a doubly ruled surface $\mathcal{S}$ that cuts the plane in $\mathcal{C}$. For $i = 1, 2, 3$, let a_i be the rules of one of its rulings that pass through A_i, and b_i the rules in the opposite ruling that contain B_i. The argument now follows verbatim the one we used for Pappus theorem, once the Dandelin configuration (without a_0 and b_0) is produced. □

A final classic theorem worth mentioning because it is frequently used, is the following. We should stress that the referred points are completely general, and not a harmonic quadruple, so that one more is needed.

Theorem 9. *Through five points in general position in a plane, there passes a unique harmonic curve.*

Proof. Color three of the points red and two blue. Consider two generating blue lines that cut the plane in the blue points. The three red lines incident to the red points and transversal to the chosen blue lines, generate a ruled surface that cuts the plane in a harmonic curve containing the five points.

Uniqueness has two aspects. The first is combinatorial. Once there is a chosen surface, for any 3-2 coloring of the points, there is a precise choice of blue lines that yields the exact same surface. That it does not depend on the chosen surface follows from a construction within the plane based on Pascal's theorem. We briefly outline it.

Choose a linear order for the five points. There remains to find the possible sixth point that closes a hexagon which satisfies the conclusion of Pascal's theorem. The chain of five points with four lines between them, determines one of the three Pappus points, P_1 say (the meet of the two extreme lines). Of the other two Pappus points, there are lines already defined on which we know they must lie. So that lines through P_1 give the other two possible Pappus points by intersection; and from them, the corresponding sixth point can be found. This parametrizes the curve with lines about P_1. □

5. Loose ends on axioms and projectivities

The development of projective geometry and harmonic curves presented above is based on a simple set of axioms that purposely are not stated explicitly at the onset. This is so because developing a theory from axioms is always very technical and tedious and excludes the intuitive graphical richness that a less technical presentation allows. We believe that a much better understanding of the subject matter may be acquired from a non technical presentation illustrated with figures that may not be indispensable but assist the mental process of the reader, just as it happens in Euclid's Elements with euclidian geometry.

The axioms on which our presentation is based are these:

(1) Any two points A and B uniquely determine a line $A \vee B$.
(2) Let A, B, C, D be four distinct points. If lines $A \vee B$ and $C \vee D$ have a common P then $A \vee C$ and $B \vee D$ also have a common point Q.
(3) There are at least two lines that do not intersect (i.e. the space is at least three dimensional).
(4) Lines have more than two points.
(5) The axiom of double-ruling.

Figure 12. Axiom 2)

They are an evolution of those suggested by John Stillwell, the main difference being the replacement of Pappus' theorem for the axiom of double-ruling. Axioms 1) and 2) are the *incidence* axioms. Axiom 3) means space is at least three dimensional. Axiom 4) is required for geometry to be non trivial. Finally, axiom 5) is the required additional principle for geometry to be rich enough as to have a deep relation to other branches of mathematics like analysis and topology.

From these axioms *planes* can be defined as all points belonging to the lines generated by a fixed point and the points of a fixed line. The incidence properties of planes are obtained from this definition and the axioms. Axiom 2) tells us then that the lines $A \vee B$ and $C \vee D$ are coplanar and excludes the possibility of non-intersecting coplanar lines (euclidian parallelism). This is probably the hardest idea in projective geometry for students to accept because it forces them to conceive some ideal points and lines "at infinity" which have no graphical representation. We believe that Axiom 2 may contribute to overcome this difficulty since it makes it unnecessary to mention parallel lines in the classical sense since parallelism is defined in projective geometry in relation to one particular line (or plane) that is taken as "the horizon".

Since it is known that the axiom of double-ruling is equivalent to Pappus' theorem, the natural aritmetization of projective space generates a commutative field. This commutativity can be broken by defining a projective space via a non-commutative field like the quaternions.

A *projectivity* is defined as the composition of projections each one sending points in one line to the points in a second line. Projectivities then constitute an algebraic structure known as *grupoid*. It is well known that Pappus' theorem is equivalent to the *axiom of three on three*, which states that a projectivity is completely determined by it's effect on any three points of it's domain. Therefore our presentation of projective geometry is equivalent, from the logical point of view to the sometimes preferred presentation based on projectivities and on the axiom of three on three (for example, Coxeter).

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