Harmonic curves and the beauty of Projective Geometry

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The purpose of this paper is to present projective geometry in a synthetic, visual and intuitive style through the central notion of *harmonicity* and in particular the exploitation of *harmonic curves*. Some interesting new results are encountered in the process, in particular what we refer to as the "Polarity Theorem" which provides simple synthetic and intuitive proofs of several classic results, like the equivalence of the groups of hyperbolic and harmonic reflexions.

1 Introduction

A *conic curve* or *section* is defined as the intersection of a *circular cone* with a plane. The concept of conic section is inherently metric i.e., non-projective. However, it is well known that conic curves somehow belong to projective geometry. This incongruent state of affairs lead us to define *harmonic curves* in purely projective style and study their properties without any reference to metric concepts. Yes, harmonic curves become conic sections in euclidean space, but they have such a rich set of purely projective properties that their study constitutes a complete and beautiful approach to projective geometry. Such approach is what this paper is all about.

Motivated by the technics of perspective drawing, developed by the painters of the renaissance to represent realistic three dimensional scenes on flat canvases, Girard Desargues (1591 - 1661) initiated the development of Projective Geometry expanding the concept of space to include *ideal points* (also known as points *at infinity*). Mostly forgotten through the XVII and XVIII centuries, his visionary work was revived and recognized as fundamental during the first half of the XIX century, when Projective Geometry was firmly established as a field on its own. In 1872 Felix Klein opened his famous *Erlangen Program*, [7], with the statement *"Among the advances of the last fifty years in the field of geometry, the development of Projective Geometry occupies the first place"*.

One of the mathematicians whose work deserved such praise is Karl Georg Christian von Staudt (1798 – 1867). In his treatise on the subject [10], he proves that the notions of *harmonicity*, i.e. the concepts of harmonic sets and pencils, are entirely projective, i.e., independent of metric ones like distances or angles. He also shows that conic curves may be defined through the purely projective concept of *polarity*, which is intimately related to harmonicity.

In an attempt to make these facts accesible to high-school teachers, and inspired by [8] where John Stillwell argues that such goal is not only worthwhile and culturally urgent, but also feasible, the authors of this article developed a *dynamic geometry* system, *ProGeo3D* [2], specialized in projective geometry. In particular, it incorporates harmonicity as a construction tool. Playing with it we found several interesting constructions, proofs and new results results that do not seem to be included in the known literature on the subject. This paper presents and uses them to propose an alternative approach to projective geometry which is intuitive, synthetic and, in our subjective opinion, beautiful.

We start by reviewing von Staudt's definition of harmonicity and emphasizing the duality it carries intrinsically. That leads us to a simple definition of what we call *armonic curves* to differentiate them from the classic treatment of conic sections, although in euclidean space they are the same. In the following section, we relate them to von Staudt's definition, which uses polarity, but now it is stated as a theorem, the "Polarity Theorem", which extends von Staudt's result to 3D situations and provides simple synthetic proofs of classic results in projective and hyperbolic geometries. The proof of the Polarity Theorem we give uses the classic idea in Projective Geometry of going out to 3D, finding a way to work out things there, and coming back to 2D. In this case, the first to use our explicit technic was Germinal Pierre Dandelin (1794 – 1847) in [4] to prove Pascal's Hexagon Theorem. It is deeply related to the ruled surfaces defined by Hilbert and Cohn-Vossen in [5] using only incidence geometry. This suggests the formulation of a new axiom for Projective Geometry, which we call the Equipal Axiom, and which is shown to be equivalent to Pappus Theorem and thus to other equivalent axioms frequently used in projective geometry. In the final section we discuss the axiomatic foundations of our approach in which the use of 3 dimensions is fundamental.

2 Triangles, quadrangles and harmonicity

A *triangle* is defined as a set of three points called its *vertices* and the three lines that incide on them, called its *sides*. Two triangles are said to be in *polar perspective* if there is a one to one relation between their vertices so that the three lines joining corresponding vertices are congruent. In a dual manner, two triangles are said to be in *axial* perspective if there is a one to one relation between their sides such that the intersections of their corresponding sides are collinear. The best known result of projective geometry is:

Theorem 1 (Desargues Theorem)**.** *Two triangles are in polar perspective if and only if they are in axial perspective.*

There is a very simple proof of this theorem when the two triangles are not coplanar, which we illustrate in see Figure 1.a. The central idea is that the planes defined by both triangles intersect in a line and this line is where corresponding sides meet. We omit the details. The proof for coplanar uses the "trick" of generating a 3D case which implies the 2D result, and is illustrated in Figure 1.b. Details are straight forward and we omit them.

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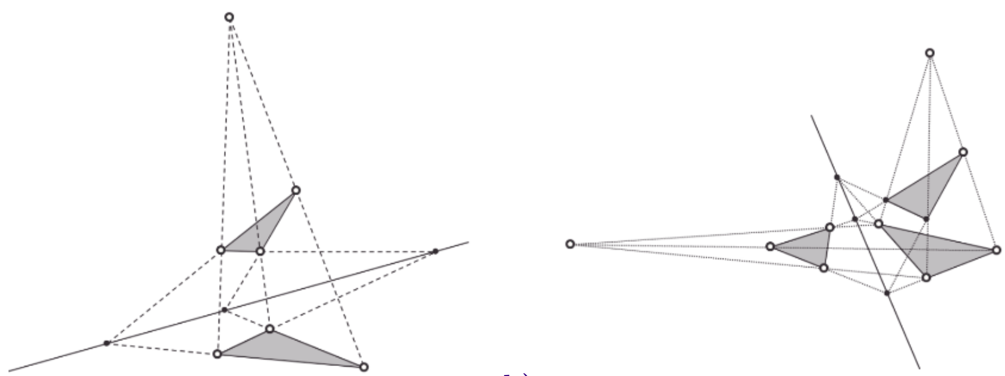

a) b)

Figure 1: **a)** Proof of Desargues Theorem in 3D. **b)** Proof of Desargues Theorem in 2D by a 3D-lift.

Quadrangles are much richer than triangles and constitute the basis for our approach to Projective Geometry. A *quadrangle* is defined as a set of four coplanar points *in general position* (i.e., no three of them being collinear), called its *vertices*, together with four lines, called its *sides*, each one joining two and only two vertices, i.e. each vertex lies in exactly two sides. A pair of sides are said to be *opposite* if they have no vertex in common. A quadrangle has two pairs of opposite sides. Two vertices are called *opposite* if no side passes through both of them. There are two pairs of opposite vertices. The *diagonals* of the quadrilateral are defined to be the two lines that join opposite vertices. The intersection of the diagonals is called the *center* of the quadrangle and the line h joining the intersections of the opposite sides (which may be called the diagonal points) is called its *horizon*. Four points in general position are the vertices of exactly three different quadrangles. Their corresponding centers and horizons form its *diagonal triangle*, which is common to all three quadrangles.

In other words, a quadrangle is defined by its vertices (4 points in general position) together with a *dihedral* order (each object of one type is incident with two of the other) on them which yields the 4 sides. The notion of quadrangle is autodual, it may also be considered as a set of three sides with a dihedral order on them. The use of the name "quadrangle" instead of "quadrilateral" is appropriate because it emphasizes this duality: the two lines which incide on each vertex (which form an "angle") distinguish two *adjacent* vertices and this also determines the *opposite* ones. Observe that to choose one of the three posible quadrangles that our points or lines may define, one has to take a combinatorial decision: to choose a dihedral order of the four points (or lines); or alternatively, one can take a geometric decision: to choose a center and a horizon. In any case, the dihedral partition into opposite pairs is what determines which of the three possible quadrangles is being defined. We use the "horizon" and "center" terminology meaning always a line and a point, respectively. In perspective drawing, these terms fit naturally because a square tile is drawn as a quadrangle whose center and horizon correspond to center of the tile and the horizon of the tiled plane to which it belongs, respectively.

One of the seminal contributions of Karl von Staudt was to prove that *harmonicity* (the notion of harmonic conjugates which had been used since antiquity in terms of cross ratios of distances) depends only on incidence relations. We now define harmonicity using only incidence and quadrangles.

Four collinear points A, C, B, D (as in Figure 2.a are said to be a *harmonic set*[1], if there exists a quadrangle such that its diagonal points are A and B and the other pair, C and D, are incident with the diagonal lines. Dually, four concurrent lines are called a *harmonic pencil*[2] if there exists a quadrangle such that one pair of lines are the diagonal lines of the quadrangle, (also called its *center*) and the other two lines are incident to the diagonal points of the quadrangle, see Figure 2.b.

[1]The terms "harmonic quadruple" or "harmonic range" are also used, but we stick to "harmonic set" as in the classic texts [9] and [3].

[2]The term "harmonic set of lines" is also used, e.g. [9, 3]; but we will use "pencil" for simplicity and to distinguish them immediately from harmonic sets (of points).

As stated, the pairs of elements in the definitions play a different role but, as we will see, they are interchangeable, so that both notions include an explicit dihedral order of the four elements involved, which coincides with their geometric placement (points within a projective line or lines about a point).

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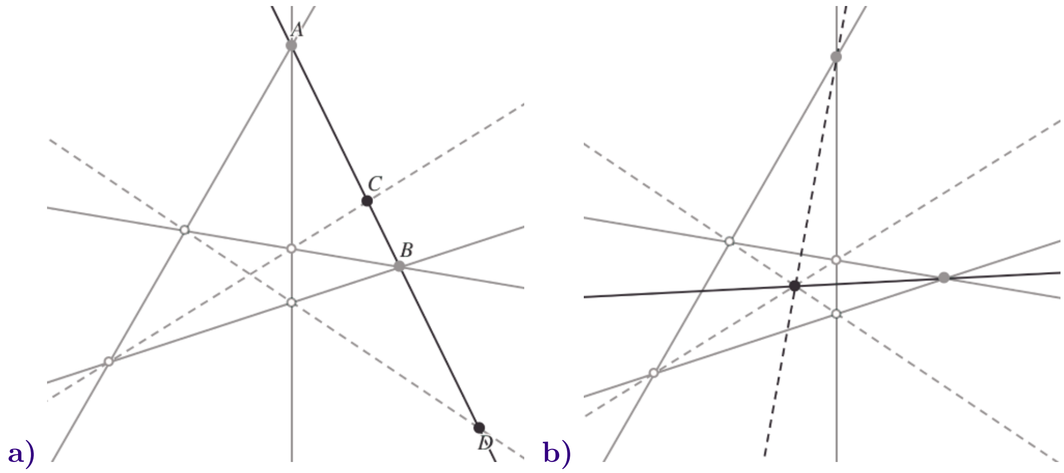

Figure 2: **a)** A harmonic set. **b)** A harmonic pencil.

Let us now show that these definitions are sound. Given a collinear triple A, C, B with C distinguished, two auxiliary points out of the support line and collinear with C determine a unique quadrilateral $\mathcal{Q}$ as in Figure 2.a, and therefore produce the point D as the intersection of the other diagonal line with the horizon; this construction, called the *harmonic fourth*, has as outcome the point D, called the *harmonic conjugate* of C with respect to A and B. Since for the triple A, D, B, one can choose the other opposite pair of vertices of $\mathcal{Q}$ as auxiliary points and then obtain C as outcome, we can further say that the (unordered) pair of points C, D are *harmonic conjugates* with respect to A, B, [3, 9].

Theorem 2 (Harmonic Theorem)**.** *The outcome of the harmonic fourth construction does not depend on the choice of the auxiliary points.*

The proof is well known and follows from Desargues' Theorem. It may also be directly proved in 3D using only incidence arguments (see Figure **??** a), and then the proof is easily extended to the planar case by a simple "3D-lifting" argument. We omit the details for brevity.

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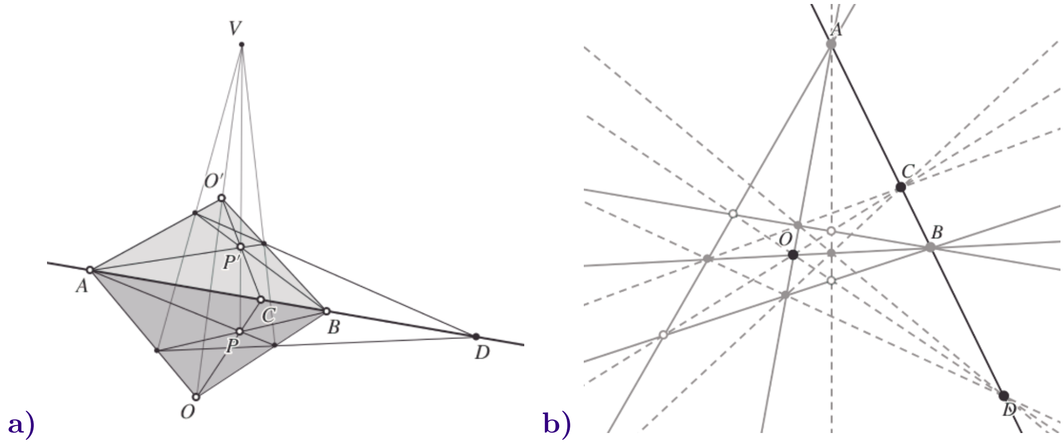

Figure 3: **a)** Visual proof of the harmonic theorem in 3D. **b)** Symmetry of harmonicity: $D = C\rho_{A,B}$ if and only if $B = A\rho_{C,D}$.

Finally, to see that the definition of harmonic set is symmetric with regard to the role played by the pairs of points, we extend the quadrangle by drawing lines from C to $(O \vee A) \wedge (P \vee B)$ and $(O \vee B) \wedge (P \vee A)$ (see Figure 3.b). Then the quadrangle with diagonals $O \vee A$ and $O \vee B$ provides a construction of the fourth harmonic A of B with respecto to C, D. □

Since the point O in Figure 3.b may be chosen to be any point not on the support line of the harmonic set, we obtain that any such point *sees* them as harmonic, that is, the lines to them with their dihedral order is a harmonic pencil. Dually, there is also a *harmonic fourth* construction for lines and any line not through the center of a harmonic pencil cuts it in a harmonic set. Thus, harmonic sets and pencils are preserved by projections.

The harmonic fourth construction makes sense in the singular cases when $C = A$ or $C = B$, since the outside quadrangle collapses to a line, but not so the construction: it holds in the sense of not becoming ambiguous, and yields $D = A$ and $D = B$, respectively. So that given two (distinct) points A and B in a line ℓ we get a well defined map

$$\rho_{A,B} : \ell \to \ell$$

called the *harmonic reflection* of ℓ with respect to A and B: it fixes these two points and it gives the harmonic conjugate elsewhere. $\rho_{A,B}$ is an involution which interchanges the two segments in which the points A and B divide their projective line. And in particular, it interchanges its ideal point at infinity with the (euclidean) midpoint of A and B, making the harmonic fourth construction a very useful tool for perspective drawing.

The natural generalization to the projective plane (space) is the *harmonic reflection* $\rho_{C,m}$ with respect to a point C, called the *center*, and a non-incident line (resp., plane) m, called the *mirror*. [3] It is defined for every point $X \neq C$ as:

$$X \cdot \rho_{C,m} = X \cdot \rho_{C,(X \vee C) \wedge m}$$

Note: we choose to write the action of maps or functions on the right of the object on which they are applied. This notion amalgamates two classic euclidean examples: the central inversions, when the mirror is the line (plane) at infinity, and the reflections when the center is the ideal point in the direction perpendicular to the mirror.

Harmonic reflections are *collineations* (i.e., they send lines to lines). They act in the dual plane as harmonic reflections in the sense that if ℓ is a line different from the mirror m, then $\ell, m, \ell \cdot \rho_{C,m}, (\ell \wedge m) \vee C$ is a harmonic pencil centered at $\ell \wedge m$. See Figure 4.a.

Lemma 1 (Klein's Triangle)**.** *Given a triangle ABC with respective opposite sides abc, then $\{\, id_{\mathbb{P}^2}, \rho_{A,a}, \rho_{B,b}, \rho_{C,c} \,\}$[4] is a group (called the Klein four-group).*

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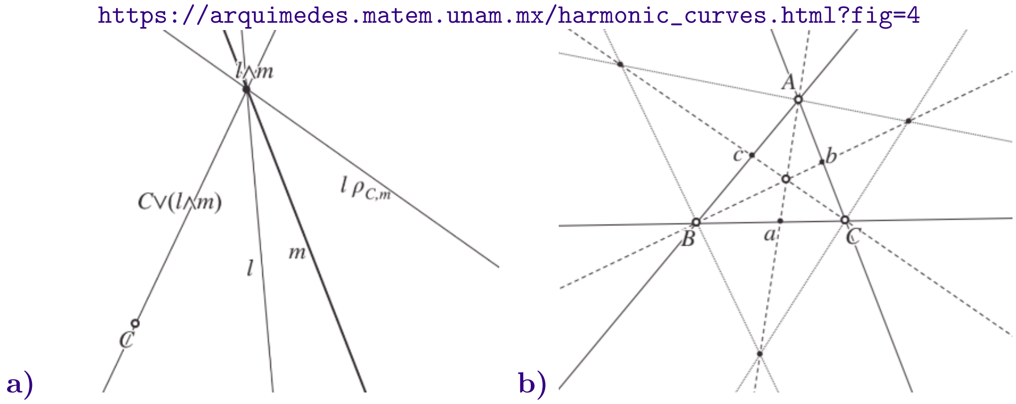

Figure 4: **a)** Harmonic pencil formed by the harmonic reflexion $\rho_{C,m}$ of a line ℓ with center C and mirror m. **b)** Klein's Triangle Lemma.

Proof. Since the three non-trivial elements are involutions, we need only to show that the composition of any two of them gives the third, which is the definition of the Klein four-group. Consider a point X not in the triangle. We claim that the quadruple $\{\, X, X \cdot \rho_{A,a}, X \cdot \rho_{B,b}, X \cdot \rho_{C,c} \,\}$ has ABC as its diagonal triangle. In Figure 4.b, the three dashed lines through X have harmonic sets that define the corresponding three points other than X. The gray lines from a vertex (say A) to one of them (say, $X \cdot \rho_{C,c}$) pass through

[3] In the plane, Coxeter calls it *harmonic homology* in [3].

[4] $id_{\mathbb{P}^2}$ stands for the identity map in the projective plane $\mathbb{P}^2$.

another one $(X \cdot \rho_{B,b})$ because the two corresponding harmonic sets (in $C \vee X$ and $B \vee X$) are projected to each other from the vertex (A) and projections preserve harmonicity.

It is easy to see that these gray lines through the vertices cut the opposite side in its corresponding harmonic conjugate, and that for points X in the triangle the maps behave as they should. This completes the proof. □

Thus, the generic orbits of the Klein four-group associated to a triangle are the quadruples that have it as diagonal triangle, and any of the four triangular regions in which the three lines cut the projective plane are the fundamental regions of the group action which has the vertices as fixed points.

We define the *harmonic group* $\mathcal{H}ar(n)$ $(n = 1, 2, 3)$ as the group of transformations generated by the harmonic reflections on $\mathbb{P}^n$.

3 Harmonic curves and bundles

We define the *harmonic curve*, $\mathcal{C}_\mathcal{Q}$ of a quadrangle $\mathcal{Q}$ as the locus of points that are centers of harmonic pencils *transversal* to $\mathcal{Q}$. By transversal we mean that each line of the pencil is incident to a vertex of $\mathcal{Q}$ and such correspondence preserves the dihedral orders. Dually, the *harmonic bundle* of a quadrangle consists of the lines that support a harmonic set transversal its sides, with corresponding dihedral orders.

Consider a quadrangle $\mathcal{Q}$ with vertices A, C, B, D. First observe that the vertices are points of its harmonic curve $\mathcal{C}_\mathcal{Q}$. Indeed, for each vertex, the harmonic conjugate of its diagonal with respect to its sides completes a harmonic pencil centered at it which is transversal to $\mathcal{Q}$, see Figure 5.a. These new lines are the *tangent* lines to $\mathcal{C}_\mathcal{Q}$ at the vertices and will be denoted by the corresponding lower case letter. The harmonic bundle of the quadrilateral a, c, b, d is called the *tangent bundle* of $\mathcal{C}_\mathcal{Q}$ and will be denoted $\mathcal{C}^*_\mathcal{Q}$.

Now consider a point $Z \in \mathcal{C}_\mathcal{Q}$ different from the vertices, we call it *generic*, see Figure 5.b. By definition, the four lines from Z to the vertices are a harmonic pencil centered at Z. Let $X = (A \vee B) \wedge (C \vee Z)$ and $Y = (A \vee B) \wedge (D \vee Z)$. Then A, X, B, Y is a harmonic set. Observe that we can recover Z from $X \in A \vee B$ by defining

$$Y = X \cdot \rho_{A,B} \quad \text{and} \quad Z = (C \vee X) \wedge (D \vee Y)\,. \tag{1}$$

But this makes sense for X ranging over $A \vee B$ and gives the four vertices, so that $\mathcal{C}_\mathcal{Q}$ is parametrized by $X \in (A \vee B)$ via this construction. We call this the *HC-construction.*

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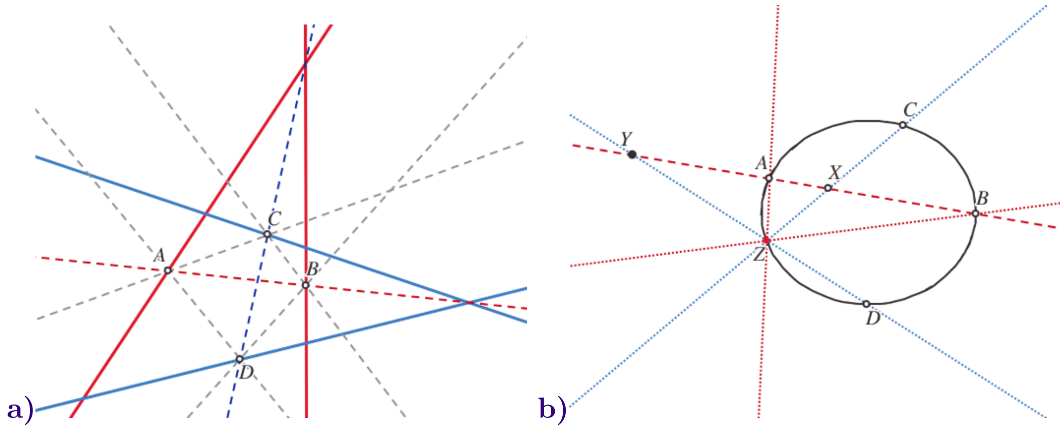

Figure 5: **a)** Quadrangle $\mathcal{Q}$ with tangent lines to its harmonic curve $\mathcal{C}_\mathcal{Q}$ at its vertices. **b)** Generic point $Z \in \mathcal{C}_\mathcal{Q}$ where A, X, B, Y is a harmonic set in $A \vee B$.

Lemma 2 (Duality lemma)**.** *Points in the harmonic curve* $\mathcal{C}_\mathcal{Q}$ *are* paired *(i.e., in bijective correspondence) by incidence with lines in its tangent bundle* $\mathcal{C}^*_\mathcal{Q}$.

Proof. Let us continue with the notation above, so that a, c, b, d, is the quadrilateral whose harmonic bundle is $\mathcal{C}^*_\mathcal{Q}$. As before, these four *generating* lines belong to the bundle because the vertex to which they are tangent (called their *contact point*) can be obtained as the harmonic fourth of their intersection to the other three lines (see Figure 5.a). Going further on the *HC*-construction (1), and dualizing it (see Figure 6): let $Q = a \wedge b$, $x = Q \vee Y$ and $y = Q \vee X$, so that a, x, b, y is generically a harmonic pencil centered at Q. Then, $z = (c \wedge y) \vee (d \wedge x)$ is a line of the bundle $\mathcal{C}^*_\mathcal{Q}$, and any such line is uniquely expressed in this way.

To prove that $Z \in z$ for X different from A and B, define $q = A \vee B$ and consider the triangle QXY with respective opposite sides qxy. By the definitions, we have $D = C \cdot \rho_{Q,q}$ $(d = c \cdot \rho_{Q,q})$, and by Klein's Triangle Lemma, $\rho_{X,x} = \rho_{Q,q} \cdot \rho_{Y,y}$, then $C \cdot \rho_{X,x} = D \cdot \rho_{Y,y}$ $(c \cdot \rho_{X,x} = d \cdot \rho_{Y,y})$. But $C \cdot \rho_{X,x} \in C \vee X$ and $D \cdot \rho_{Y,y} \in D \vee Y$, so that $C \cdot \rho_{X,x} = (C \vee X) \wedge (D \vee Y) = Z$ (dually, $c \cdot \rho_{X,x} = z$). Hence, the fact that $C \in c$, implies that $Z \in z$ as we wished. Z is called the *contact point* of $z \in \mathcal{C}^*_\mathcal{Q}$ which is the *tangent line* to $\mathcal{C}_\mathcal{Q}$ at Z. □

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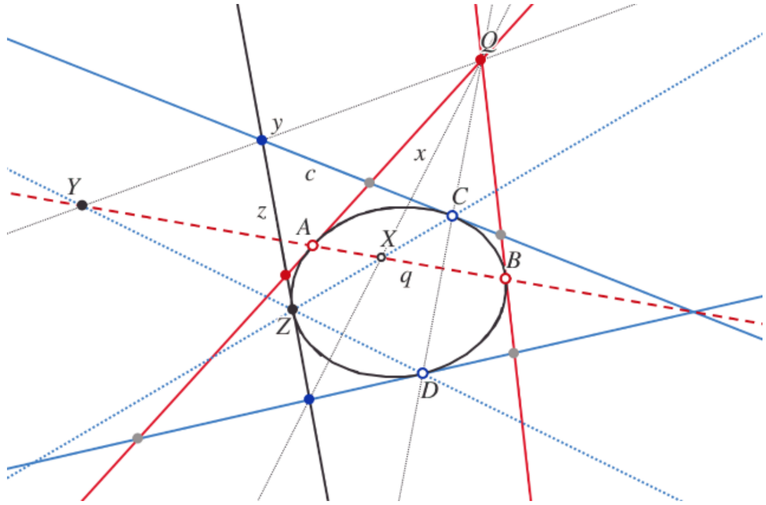

Figure 6: Incidence of points in a harmonic curve and lines in its tangent bundle.

As a corollary, we can express the harmonic curve $\mathcal{C}_Q$ as a family of harmonic reflections applied to a single point

$$\mathcal{C}_Q = \{\, C \cdot \rho_{X,x} \,|\, X \in (A \vee B) \setminus \{A, B\}\,\} \cup \{A, B\}\,, \tag{2}$$

where $x = (X \cdot \rho_{A,B}) \vee (a \wedge b)$, which only depends on three points A, C, B and the two tangent lines a, b incident to A, B, respectively; we will call this, the *A-construction*.

Clearly, harmonic curves are sent to harmonic curves under projections because projections preserve harmonicity. So that the fact that the classic conic sections are harmonic curves follows from the fact that a circle is a harmonic curve. Indeed, consider an inscribed square to a circle as its generating quadrangle and use the inscribed angle theorem to prove that each of its points is the center of a transversal pencil to the vertices of the square.

4 Polarities and hyperbolic geometry

A *polarity* in the plane (in space) is a bijective correspondence between points and lines (planes) that preserves incidence; the terms *polar* of a point, *pole* of a line (plane) or a *polar pair* are used[5].

Theorem 3 (Polarity)**.** *A harmonic curve $\mathcal{C}$ induces a polarity (expressed by upper and lower case of the same letter) satisfying:*

[5]An extra hypothesis is required in [3]. Namely, that for some line, the map to the line pencil of its pole be a projectivity. But we do not need to stress this issue.

i) $P \in \mathcal{C} \Leftrightarrow P \in p$.

ii) *If* $P \notin \mathcal{C}$ *then the harmonic reflection* $\rho_{P,p}$*, with* P *as center and its non-incident polar line* p *as mirror, leaves* $\mathcal{C}$ *invariant.*

We have already seen a part of item (i) as Lemma 2 because tangent lines to a harmonic curve are defined as their polar lines. The rest of the proof will be given in the next section as a consequence of a more general result, the*Polar Theorem* in 3D. For the moment, let us make two remarks about this theorem and, assuming it is true, explore some of its profound consequences.

Two mathematicians directly associated to this theorem are Jean-Victor Poncelet (1788 - 1867) and Karl G. C. von Staudt (1798 - 1867). Poncelet proved the relation of poles and polars for conic sections using harmonicity (in its metric version), and soon after, von Staudt developed polarities as a general concept and used it as an alternative way to define conic curves within projective geometry with no metric or algebraic considerations, [10]. This definition via polarities is the one Coxeter uses in his influential book [3], and calls it "extraordinarily natural and symmetrical" because it has duality built into it. In general, there are two types of polarities: *euclidian* in which no point is incident with its polar line, and *hyperbolic* when there exist pole and polar incident pairs. The terms used are related to the groups generated by harmonic reflections of non-incident polar pairs. So that von Staudt's definition of a conic curve is equivalent to item (i) of the theorem for a hyperbolic polarity, while Poncelet's results can be rephrased as item (ii).

As examples of polar pairs, we have named lines and points in Figure 6 according to the upper and lower case rule for poles and polars with respect to the displayed harmonic curve $\mathcal{C}_{\mathcal{Q}}$. Indeed, a point $P \in \mathcal{C}_{\mathcal{Q}}$ and the corresponding line in $p \in \mathcal{C}^*_{\mathcal{Q}}$ described in Lemma 2 constitute a polar pair satisfying (i).

We now prove that von Staudt's definition of conic curves with mild extra hypothesis gives harmonic curves.

Lemma 3. *Given a polarity in the plane, let* $\mathcal{C}$ *be the set of points that are incident to their polar line and suppose item (ii) of Theorem 3 holds. If every line meets* $\mathcal{C}$ *in at most two points and* $\mathcal{C}$ *contains at least three points, then* $\mathcal{C}$ *is a harmonic curve.*

Proof. Let $A, B, C \in \mathcal{C}$ be three points. By the hypothesis on the lines, they form a triangle. Let a, b be the respective polar lines of A, B, so that $A \in a$ and $B \in b$. Let $Q = a \wedge b$; it is the pole of $q = A \vee B$ because polarities preserve incidence, which also implies that $Q \notin q$. Finally, let $\mathcal{Q}$ be the quadrangle $A, C, B, D = C \cdot \rho_{Q,q}$. To conclude the proof we will show that $\mathcal{C} = \mathcal{C}_{\mathcal{Q}}$.

Given $X \in q \setminus \{ A, B \}$, its polar x is a line through Q different from a and b. Let $Y = x \wedge q$. Then, since $\rho_{X,x}$ leaves q and $\mathcal{C}$ invariant and $q \cap \mathcal{C} = \{ A, B \}$, it transposes A and B, so that X, A, Y, B is a harmonic set. Since the polarity satisfies (ii), $C \cdot \rho_{X,x} \in \mathcal{C}$, so that the A-construction (2) for $\mathcal{C}_\mathcal{Q}$ implies that $\mathcal{C}_Q \subset \mathcal{C}$. Finally, given $Z \in \mathcal{C}$ different from A, B, C, let $X = (Z \vee C) \wedge q$, then $Z = C \cdot \rho_{X,x}$ because the line $Z \vee C$ has no point in $\mathcal{C}$ other than Z and C by hypothesis. Therefore, $\mathcal{C}_Q = \mathcal{C}$. □

One very interesting consequence of the Polarity Theorem is that it provides the projective model of the hyperbolic plane due to Beltrami and Klein.

Let $\mathcal{C}$ be a harmonic curve. If we assume the Polarity Theorem, the curve necessarily comes with a polarity. The curve breaks the plane into two regions. The *interior* points defined as those whose polar line does not intersect $\mathcal{C}$ form the *hyperbolic plane*, denoted $\mathbb{H}^2$. The *exterior* points are defined as those whose polar line cuts $\mathcal{C}$ in two points. Their intersections with $\mathbb{H}^2$ are the *hyperbolic lines*, which may also be considered as projective lines. The remaining projective lines are the *tangents* that touch $\mathcal{C}$ only at their pole or *contact point*.

Given a hyperbolic line q, with pole Q, let $\eta_q = \rho_{Q,q}$ be the *hyperbolic reflection* along q which maps $\mathcal{C}$ to itself by item (ii) and thus it also maps $\mathbb{H}^2$ to itself. All the hyperbolic reflections generate the group $\mathcal{H}yp(2)$ of hyperbolic transformations which, in the spirit of Klein's Erlangen Program [7], acting on $\mathbb{H}^2$ yields the hyperbolic plane geometry.

Two hyperbolic lines q and p are *perpendicular* if the hyperbolic reflection on one of them leaves the other invariant, that is, if $p = p \cdot \eta_q$. This happens iff the pole of q is incident with p. We can always write $q = A \vee B$ with $A, B \in \mathcal{C}$ and $p = C \vee D$ with $C, D \in \mathcal{C}$. Then p and q are perpendicular if and only if the quadrangle A, C, B, D is a *generating quadrangle* of $\mathcal{C}$, that is, it has $\mathcal{C}$ as its harmonic curve.

Theorem 4. *$\mathcal{H}yp(2) \cong \mathcal{H}ar(1)$.*

Proof. First, we define the *tangential map* from $\mathcal{C}$ to a tangent line, Figure 7.a. Let T be a point in $\mathcal{C}$ and let t be its tangent (or polar) line. For every $X \in \mathcal{C}$ other than T, let $X' = t \wedge x$, where x is the tangent line to $\mathcal{C}$ at X. Taking $T = T'$, this gives a bijective map $X \leftrightarrow X'$ between $\mathcal{C}$ and t, because x' (the polar of $X' \in t$) cuts $\mathcal{C}$ in T and X for $X' \neq T$.

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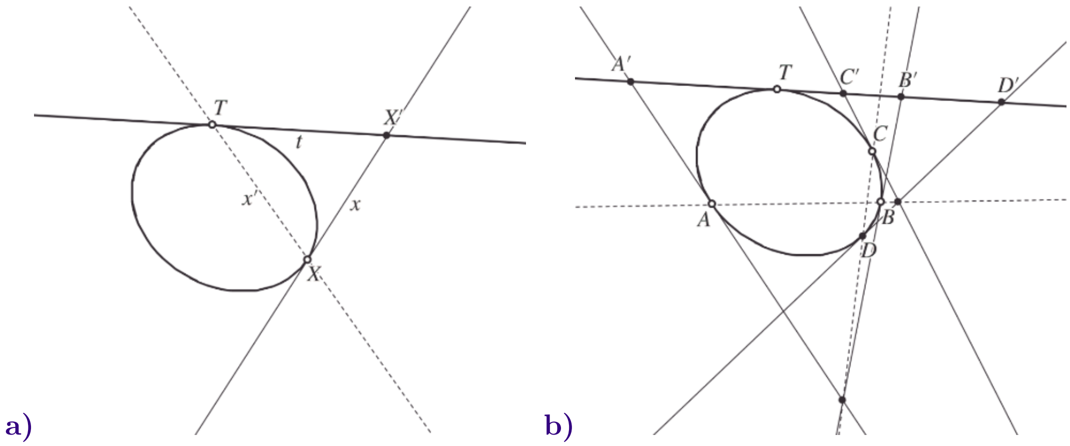

Figure 7: **a)** Tangential map. **b)** A generating quadrangle in $\mathcal{C}$ and its corresponding harmonic set in p.

Considering t as $\mathbb{P}^1$, the theorem follows from the fact that generating quadrangles of $\mathcal{C}$ and harmonic sets of t correspond to each other, because then harmonic reflections and hyperbolic reflections (the generators of the groups) correspond under the tangential map.

Let A, C, B, D be a generating quadrangle of $\mathcal{C}$, and let a, c, b, d be their respective tangent lines. By Lemma 2, the harmonic bundle of this quadrilateral is the tangent bundle of $\mathcal{C}$ and it contains t. Therefore, by the definition of harmonic bundles, we have that A', C', B', D' is a harmonic set, see Figure 7.b. □

This proof was the motivation for our definition of *harmonic curves*.

5 Doubly ruled surfaces

Our proof of the Polarity Theorem (3) is inspired by Dandelin's proof of Pascal's Hexagonal Theorem. Given a conic curve, Dandelin constructs, in [4], a hyperboloid of revolution that has it as a plane section; then, using the fact that this surface is doubly-ruled, he obtains a configuration of 6 lines in three dimensional space associated to the six points of the hexagon in the conic, and argues with the geometric-combinatorial properties of the configuration to conclude the proof. We use the same general idea and get to the same configuration of 6 lines, but instead of hyperboloids of revolution we can now use general ruled surfaces following Hilbert and Cohn-Vossen's construction of ruled surfaces in [5], which appeared in print almost a century after Dandelin's proof, and made clear that they can be constructed by simple incidence arguments.

Consider two lines a and b in three dimensional projective space. They touch if and only if they are coplanar. If this is not the case, they can be called a *generating* pair because for any point X not in them, there exists a unique line through X *transversal* (i.e., with a common point) to a and b; in fact:

$$(X \vee a) \wedge (X \vee b)\,.$$

Now consider three lines a, b, c in *general position* (i.e., each pair is generating, or equivalently, no pair of them is coplanar). The *transversal ruling* to a, b, c, denoted $\mathcal{R}(a, b, c)$, is the set of lines that are *transversal* to them (i.e., that touch all three); any such set of lines will be called a *ruling* and its elements are called its *rules* (see Figure 9.a. If we denote $\mathcal{R} = \mathcal{R}(a, b, c)$, the above observation implies that $\mathcal{R}$ is parametrized by incidence with the points in any of the three generating lines (through any point in them there passes a unique rule). It will be important to note that, dually, $\mathcal{R}$ is also parametrized by planes containing one of the lines; if we denote planes by greek letters (points and lines are, respectivelly, upper and lower case latin) we have, for example, that

$$\mathcal{R}(a, b, c) = \{\, (b \wedge \alpha) \vee (c \wedge \alpha) \,|\, a \subset \alpha \,\}\,. \tag{3}$$

Every pair of rules in $\mathcal{R}$ is generating, otherwise, their three transversal lines a, b, c would be coplanar. Thus, for any triplet $a', b', c' \in \mathcal{R}$ we get a transversal rulling $\mathcal{R}(a', b', c')$ that contains the original three lines, a, b, c; this ruling is an *extension* of a, b, c (see Figure 9.b. In *real* projective space there is only one extension to a ruling of three lines in general position, but this is not true in all projectives spaces. In what follows we will assume that our projective space does have this property, i.e. we will assume it as an axiom:

Equipal Axiom.[6] *Three lines in general position belong to a unique ruling.*

Later on we will prove it is equivalent to Pappus's Theorem and thus to other classic statements that are commonly adopted as axioms in projective geometry. Another name for it could be the "Double-ruling Axiom" because it immediately implies that rulings are matched or paired: any ruling has an *opposite ruling* which is the transversal ruling to any three of its rules. The *doubly-ruled surface* (we also refer to it simply as a *ruled surface*) obtained as the union of the rules in any such ruling is also the union of the rules in its opposite ruling.

[6] Equipal is a classic mexican style of furniture that uses double rulings for bases, [1].

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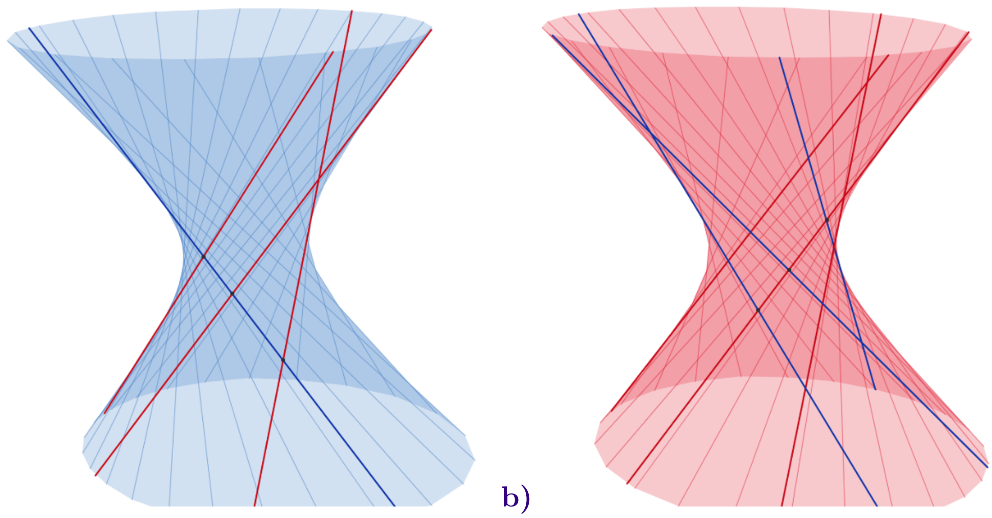

Figure 8: **a)** The transversal ruling by blue lines to three red lines. **b)** The transversal ruling to any three blue rules contains the three original red lines.

Hence, every point on a ruled surface has a *tangent plane*, the one generated by the unique rules through the point in the opposite rulings of the surface. We may see this as its *polar plane*. Actually, this association extends to a full scale polarity.

Theorem 5 (Polarity of ruled surfaces)**.** *The pairing of points in a ruled surface $\mathcal{S}$ with their tangent planes extends to a polarity of projective space. Furthermore, if $P \notin \mathcal{S}$ then P is not incident with its polar plane π and the harmonic reflection $\rho_{P,\pi}$, with P as center and π as mirror, leaves $\mathcal{S}$ invariant.*

Proof. The ruled surface $\mathcal{S}$ has two opposite rulings $\mathcal{R}$ and $\mathcal{R}'$ such that

$$\mathcal{S} = \bigcup_{x \in \mathcal{R}} x = \bigcup_{y \in \mathcal{R}'} y\,.$$

To define the polarity induced by $\mathcal{S}$ in its complement, fix three rules a, b, c in the ruling $\mathcal{R}$, and beware that we have inverted the notational use of primes: their transversal ruling is now $\mathcal{R}' = \mathcal{R}(a, b, c)$.

Consider a point $P \notin \mathcal{S}$; dually, we could start with a non-tangent plane.

Let $\alpha = a \vee P$. There is a well defined rule $a' \in \mathcal{R}'$ for which $P \in a \vee a' = \alpha$ (namely, $a' = (b \wedge \alpha) \vee (c \wedge \alpha)$, as in (3)). Let $A = a \wedge a' \in \mathcal{S}$. Observe that A must be in the polar plane of P because polarities preserve incidence and P is in the polar plane of A.

Analougously, we obtain $b', c' \in \mathcal{R}'$, for which $P \in b \vee b' = \beta$ and $P \in c \vee c' = \gamma$. Let $B = b \wedge b'$ and $C = c \wedge c'$, so that the polar plane to P has to be

$$\pi = A \vee B \vee C\,.$$

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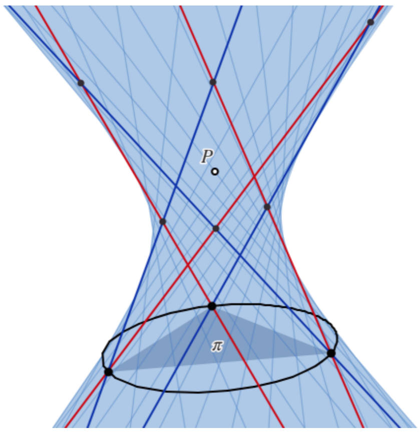
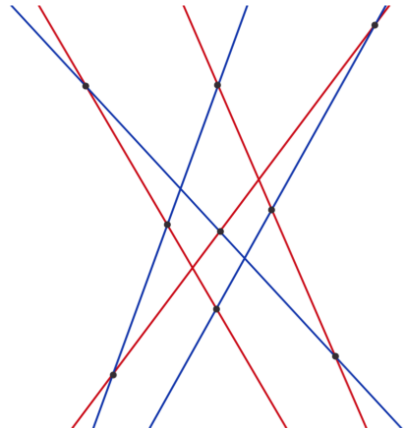


a)



b)

Figure 9: **a)** Polar plane π of pole P in ruled surface. **b)** Dandelin configuration.

If we had started, dually, with a non tangent plane π we would have found P as the intersection of the three tangent planes at $A = a \wedge \pi$, $B = b \wedge \pi$, $C = c \wedge \pi$; and a', b', c' would be the rules in $\mathcal{R}'$ passing through A, B, C respectively. So that the pairing of points and planes is now well defined.

We have distinguished what we will call a *Dandelin configuration*: six lines of two *types* or *colors*, three of each, a, b, c and a', b', c'—unprimed and primed in the text, red and blue in the pictures as in Figure 9.b— such that a pair of them touch if and only if they have opposite types. This produces nine *basic* points and nine *tangent* planes by the "meet" ($\wedge$) or "join" ($\vee$) of lines of different colors; but it also comes with a derived configuration of other lines and planes that naturally arise from them. The geometric richness of this configuration, closely related to the combinatorics of 3×3 determinants, is what Dandelin exploited in [4]; and we follow suit.

Now, we will prove that the harmonic reflection, $\rho_{P,\pi}$, with center P and mirror π interchanges the lines a, b, c respectively with a', b', c' in the opposite ruling. By the triangular symmetry of the construction, it will sufffice to prove that:

- *in the tangent plane to A, $\alpha = a \vee a'$, the lines $a, A \vee P, a', \alpha \wedge \pi$ are a harmonic pencil centered at A.*

Because this happens if and only if $\rho_{P,\pi}$ interchanges the lines a and a'.

The tangent plane $\alpha = a \vee a'$ contains five of the nine basic points of our Dandelin configuration. Namely, the α-quadrangle:

$$a \wedge b', b \wedge a', a \wedge c', c \wedge a' ,$$

with its center A because its diagonals are a and a'. The remaining four basic points outside of α, group naturally into two pairs whose generated lines are incident with the two diagonal points of the α-quadrangle. This follows because these diagonal points can be seen as the intersection of three tangent planes. Namely, $P = \alpha \wedge \beta \wedge \gamma$ and $R = \alpha \wedge (b \vee c') \wedge (c \vee b') = \alpha \wedge (B \vee C) \in \alpha \wedge \pi$ (see Figure 10).

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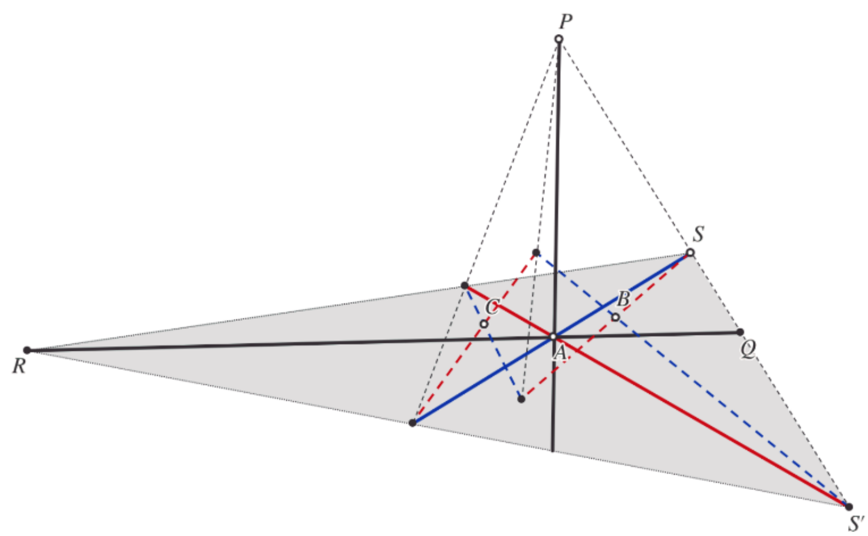

Figure 10: The Dandelin configuration given by the point $P \notin \mathcal{S}$ and its polar plane $\pi = A \vee B \vee C$ with the harmonic pencil $a, A \vee P, a', A \vee Q = \alpha \wedge \pi$.

Thus, $\rho_{P,\pi}$ interchanges the rules a and a'. Analogously, it interchanges b with b' and c with c'. Then, it gives a bijection between the transversal rulings of a, b, c and a', b', c', which are $\mathcal{R}'$ and $\mathcal{R}$ respectively, because a line transversal to a, b, c is sent by $\rho_{P,\pi}$ to a line transversal to a', b', c' and viceversa. Therefore, $\rho_{P,\pi}$ leaves $\mathcal{S}$ invariant, as we wished to prove.

In particular, since a harmonic reflection sends a line to a line concurrent with the mirror and coplanar with the center, our definition of the polarity does not depend on the choice of generating rules a, b, c.

Finally, the proof that the polarity we have defined preserves incidence follows in cases, but in a straightforward manner from the fact that if the tangent plane to a point in $\mathcal{S}$, say A as above, contains a point not in $\mathcal{S}$, say P, then the polar plane of P contains A. □

Observe that, because of the incidence invariance, the polarity extends naturally to a pairing of lines. The polar of a line ℓ is the intersection of all the polar planes of its points, or of any two of them.

This polarity theorem asserts that what one sees as the contour of a ruled surface is exactly its section with the polar plane of the viewpoint. Sections and the contour of projections coincide. We now prove that sections of ruled surfaces are harmonic curves, and that the corresponding harmonic bundle is the projection from the pole of any one of the two rulings.

Proof of Theorem 3. Consider a harmonic curve, $\mathcal{C}$, in a plane π. Our basic aim is to prove that

- *there exists a ruled surface $\mathcal{S}$ that has $\mathcal{C}$ as a section,*

that is, such that $\mathcal{C} = \mathcal{S} \cap \pi$. This will induce the desired polarity in π to complete the proof of the theorem.

By definition, $\mathcal{C}$ is the harmonic curve of a quadrangle A, C, B, D. Let a and b be the tangents at A and B, respectively; and let $Q = a \wedge b$, $q = A \vee B$. We know that $D = C \cdot \rho_{Q,q}$ and that $\mathcal{C}$ is obtained by the A-construction (2).

Choose two points P and S not in π and colinear with Q (see Figure 11.a).

Let $S' = S \cdot \rho_{P,Q}$. Since $S \neq S'$, the four lines from S and S' to A and B can be colored red and blue so that only lines of opposite colors touch. Finally, consider the red (blue) line through C transversal to the two blue (red) lines. We now have a Dandelin configuration of six lines colored red and blue: let $\mathcal{S}$ be the doubly ruled surface it defines (Figure 11.b). By construction, P and π are a polar pair with respect to $\mathcal{S}$.

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The polarity induced by $\mathcal{S}$ restricts naturally to a polarity in the plane π as follows. The polar line of a point in π is the intersection of its polar plane with π, and the pole of a line in π is the pole of the plane it generates with P —or the intersection with π of its polar line. In particular, item (i) of Theorem 3 follows for $\mathcal{S} \cap \pi$.

Since harmonic reflections preserve the planes through their center, those for non-incident polar pairs with pole in π, restrict to harmonic reflections of π that leave $\mathcal{S} \cap \pi$ invariant. Therefore, item (ii) of Theorem 3 follows for $\mathcal{S} \cap \pi$.

That $\mathcal{C} = \mathcal{S} \cap \pi$ now follows from Lemma 3 and its proof because a line that intersects $\mathcal{S}$ in three different points is easily seen to be a rule of $\mathcal{S}$ and π contains no such rules. □

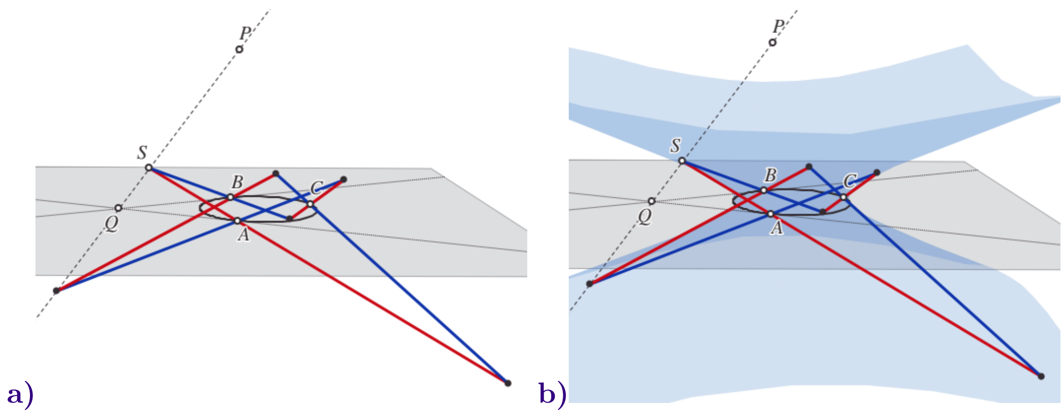

Figure 11: **a)** A Dandelin configuration arising from the input of the A-construction of $\mathcal{C}$ in a plane π. **b)** The corresponding ruled surface that intersects π in $\mathcal{C}$.

Observe that, within the above framework, for any point in $\mathcal{S} \cap \pi$ the intersection with π of its tangent plane to $\mathcal{S}$ is the projection to π from P of any of its two rules. So that we may state the following theorem as a corollary to the preceding proofs.

Theorem 6. *Harmonic curves are the sections of ruled surfaces with non-tangent planes. Moreover, harmonic bundles are the projection of rulings from external points, and the tangent bundle of a section of a ruled surface is the projection from the corresponding pole of any of its two rulings.* □

Finally, we prove the following theorem, making the appropriate remarks to acknowledge Dandelin's original proof of Pascal's Hexagon Theorem that inspired our treatment.

Theorem 7. *The Equipal Axiom is equivalent to Pappus's Theorem.*

Proof. First, we must state Pappus's Theorem:

- *The opposite sides of a planar hexagon whose vertices lie alternatively in two lines, meet in three collinear points.*

Let a_0 and b_0 be coplanar lines with points $B_1, B_2, B_3 \in a_0$ and $A_1, A_2, A_3 \in b_0$, so that the hexagon of Pappus's hypothesis is $A_1, B_2, A_3, B_1, A_2, B_3$ considered cyclically, and the theorem asserts that the three "*Pappus's points*"

$$P_i = (A_j \vee B_k) \wedge (A_k \vee B_j)\,,$$

where $\{i, j, k\} = \{1, 2, 3\}$, are collinear.

The hypothesis of Pascal's Theorem is that the six points named above lie not on two lines, but on a harmonic curve and the conclusion is exactly the same.

Dandelin's proof considers rules (a_i and b_i, $i = 1, 2, 3$) through the vertices alternatively in the two rulings of a ruled surface. For the case of Pascal, this would now follow immediately from Theorem 6; for Pappus, we need to work a little more because the plane $\pi = a_0 \vee b_0$ will turn out to be a tangent one.

Let a_1, a_2 be a pair of generating lines that meet π in A_1, A_2 respectively. Let $\mathcal{R}' = \mathcal{R}(a_0, a_1, a_2)$ so that $b_0 \in \mathcal{R}'$ and let $b_1, b_2, b_3 \in \mathcal{R}'$ be the rules through B_1, B_2, B_3 respectively. Now, let $\mathcal{R} = \mathcal{R}(b_0, b_1, b_2)$ so that $a_0, a_1, a_2 \in \mathcal{R}$ and finally, let $a_3 \in \mathcal{R}$ be the rule through $A_3 \in b_0$.

We have defined eight lines of two types or colors, a_i and b_j with $0 \le i, j \le 3$, such that all pairs of different color except one do meet, namely, a_i meets b_j for all $i \ne 3 \ne j$. The Equipal Axiom implies that $\mathcal{R} = \mathcal{R}(b_0, b_1, b_3)$ and thus, that $a_3 \in \mathcal{R}$ meets b_3. But moreover, the Equipal Axiom follows if this is always true for the general setting of eight lines, because it implies $\mathcal{R} = \mathcal{R}(b_0, b_1, b_3)$ letting a_3 run in all of $\mathcal{R}'$, and then moving the b_j's around $\mathcal{R}'$, this implies that a_0, a_1, a_2 extend to the unique ruling $\mathcal{R}$.

So, we are left to prove that the Pappus's points P_1, P_2, P_3 are collinear if and only if a_3 and b_3 meet, see Figure 12

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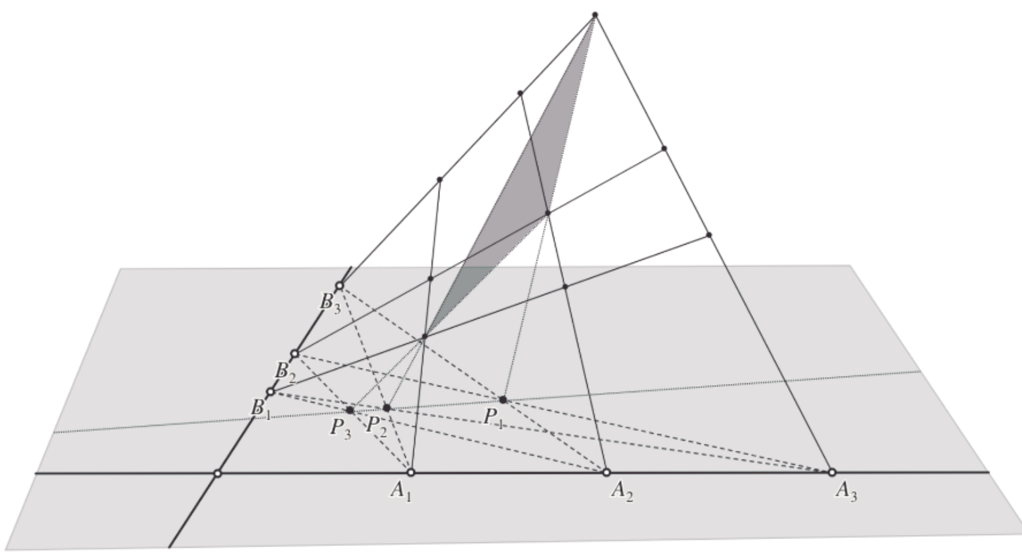

Figure 12: A Dandelin configuration over a plane with a Pappus configuration.

Suppose that a_3 and b_3 meet. Then a_i and b_j, with $i, j \in \{1, 2, 3\}$ is a Dandelin configuration. For any such i, j we have that

$$A_i \vee B_j = (a_i \vee b_j) \wedge \pi \, .$$

So that the Pappus's points may be seen as lines intersecting π:

$$P_i = ((a_j \vee b_k) \wedge (a_k \vee b_j)) \wedge \pi = ((a_j \wedge b_j) \vee (a_k \wedge b_k)) \wedge \pi \,, \tag{4}$$

for $\{i, j, k\} = \{1, 2, 3\}$. But these three lines meet pairwise, therefore they lie in a plane that defines the *Pappus's line*:

$$p = ((a_1 \wedge b_1) \vee (a_2 \wedge b_2) \vee (a_3 \wedge b_3)) \wedge \pi \,,$$

which proves Pappus's Theorem and, for non-tangent planes π constitutes Dandelin's proof of Pascal's Theorem.

We are left to prove that Pappus implies the Equipal Axiom which, as we have seen, follows from proving that a_3 meets b_3 assuming that P_1, P_2, P_3 lie in a line $p \subset \pi$. Observe that (4) still holds for $i = 3$ (and $\{j, k\} = \{1, 2\}$), so that

$$\delta = p \vee ((a_1 \wedge b_1) \vee (a_2 \wedge b_2))$$

is a plane because the two lines meet at P_3. It contains the lines

$$\ell_1 = P_1 \vee (a_2 \wedge b_2) \quad \text{and} \quad \ell_2 = P_2 \vee (a_1 \wedge b_1) \,,$$

which give us a point $W = \ell_1 \wedge \ell_2$. To see that $W \in a_3$ and $W \in b_3$ to conclude the proof, observe that W can be seen as the intersection of three planes in two ways; namely, of $\delta, (a_3 \vee b_1), (a_3 \vee b_2)$ and $\delta, (a_1 \vee b_3), (a_2 \vee b_3)$. □

6 Loose ends on axioms and projectivities.

What we have done in this paper works verbatim in the abstract setting of projective space. The included images, which are designed to help the reader to understand intuitively the abstract arguments are, of course, drawn in euclidean 2 and 3 dimensional spaces. However no euclidean arguments are used, except when showing that in euclidean space harmonic curves become the conic sections. Since Projective Geometry has always been ground for considerations about math foundations, we think it is appropriate to close the article with a few remarks concerning axiomatics and the rigorous mathematical content of our proofs.

The axioms on which our presentation is based and all its theorems are proved are the following. A *projective space* consists of a ground set, or *space*, of *points* with a well defined family of subsets called *lines*, satisfying:

1. Any two distinct points A and B lie on a unique line $A \vee B$.

2. If A, B, C, D are four distinct points and lines $A \vee B$ and $C \vee D$ meet, then the lines $A \vee C$ and $B \vee D$ also meet.
3. There are two lines that do not meet.
4. Lines have more than two points.
5. The harmonic fourth of three collinear points is neither of them.
6. The Equipal Axiom.

These axioms are a variation of those commonly used (e.g., the ones suggested by Stillwell in [8]). The main difference being the replacement of Pappus's Theorem by the Equipal (or Double-rulings) Axiom. Axioms 1 and 2 are the fundamental *incidence* axioms. The statement of Axiom 2 is attributed to Pash and Veblen; it cleverly says that two lines meet if and only if they are *coplanar* without the need of having planes previously defined. Axiom 3 means the space is at least three dimensional (more than a plane), and it is known to be equivalent to Desargues's Theorem. Axiom 4 is required for geometry to become interesting and not simply some trivialized set theory.

The two final axioms depend on some further development of the theory; they are not primitive. Axiom 5 guarantees that the *ground field* is of characteristic different from 2 or, equivalently, that the geometry does not contain the Fano Plane. The *characteristic* of a projective space can be defined geometrically using the harmonic fourth construction; essentially from how far can one go in a *harmonic sequence* without returning. It is needed here to make sense of harmonic curves (and that harmonic reflections are not the identity) because it implies that harmonic sets that have exactly four points do exist. Axiom 6 is a required additional principle for geometry to be rich enough to have a deep relation to other classic branches of mathematics like analysis and topology; bellow, we will discuss the several versions it may adopt.

From the first 4 axioms, *flats* can be defined as the *closed* subsets under the operation of taking lines, and then, the *dimension* of a flat is obtained as one less than the number of points needed to *generate* it; so that *planes* are defined as flats of dimension 2, [9]. The incidence properties of planes and lines in a space of dimension 3 are obtained from this; and the Hilbert-Cohn Vossen construction of ruled surfaces follows, making sense of the statement of the Equipal Axiom.

Since the Equipal Axiom is equivalent to Pappus's Theorem, the arithmetization of projective space yields as *ground field* a commutative one. In [8] commutativity is proved from Pappus's Theorem and the ground field is described from scratch. The necessity of the axiom is proved by constructing a projective space over a non-commutative field like the quaternions.

Another widely used version of Axiom 6 is as the uniqueness part of the Fundamental Theorem of Projective Geometry. It is usually stated in the context of planar geometry where Axioms 1 and 2 become appealingly *dual* (and Axiom 3 is false). A *projectivity* is defined as the composition of *projections* between (points in) lines or (lines in) concurrent pencils; they are always bijections. It is not hard to construct a projectivity determined by it's (arbitrary) effect on (any) three elements of it's domain. This is the existence part of the Fundamental Theorem. However, the uniqueness is proved to be equivalent to Pappus's Theorem, so one must be assumed to prove the other, see [3], [9].

We think that Axiom 3 is natural because it responds and gives credit to the original motivation for the creation of Projective Geometry which is, undoubtedly, renaissance perspective, in which dimension 3 is essential. But then, if one thinks about projectivities between non coplanar lines in a three dimensional projective space, one is naturally lead to consider ruled surfaces. Indeed, given a projectivity from a line a to a non coplanar line b, the set of lines joining a point in a to its image in b turns out to be a ruling. So, the Equipal Axiom is intimately related to the uniqueness of projectivities given by three arbitrary values (the projectivity is determined by the extended ruling of three lines). Moreover, this association of a set of lines to a function between lines is also a classic idea. It is the dual of how Jakob Steiner (1796–1863) defined conics in a purely projective manner; and is a natural, visual way of presenting them, e.g., [9].

Projective geometry is remarkable in many ways. One of them is the importance of some mathematical notions that were worked and experimented within it long before their abstract general acknowledgement. For example, projectivities were masterly used almost one century before the notion of sets and the language of abstract functions was stablished; moreover, together they constitute what we now call a groupoid (defined in the mid XX century within category theory). And of course, there is the leading role it played in broadening our notion of geometry and its influence on the dawn of topology. There are many ways to approach it and present it. We hope this paper contributes to the awareness of its cultural significance and the convenience and possibility of finding "its way down into secondary schools", [6], or at least into early undergraduate courses.

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