# A GEOMETRIC APPROACH TO PLANETARY MOTION AND KEPLER LAWS

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Abstract. We present some properties of the ellipse and use them to derive Keplers laws for planetary motion. Our arguments rely on elementary classical mechanics, in particular the conservation of energy, and angular momentum.

## 1. Introduction

Kepler showed, phenomenologically, that the planets move along ellipses and Newton derived from this that the central force must be proportional to the inverse square of the distance, [8]. Later Newton's universal law of gravity was taken as a basic principle and Johannes Bernoulli proved that in a field of central attraction where the force is proportional to the inverse square of the distance, the trajectories are always conic sections [9]. Concerning Newton's *Principia* we refer also to [2] and [5].

This masterpiece of human achievement was retold many times and several different approaches have been published, see for example [1, 3, 4, 6] or [7]. Our modest contribution is aimed at providing a somewhat deeper understanding of the intimate relations that exist between the geometric and the physical aspects of planetary and projectile motion and to provide an alternative treatment of this subject which may be of value in some cases.

**Question.** *Given the location $O$ of the atractive force $\vec{F}$, the position $P$ of the mass $m$ and the initial velocity $\vec{v}$, how can the trajectory of the object be constructed?*

It is assumed that the force is given as

$$\vec{F} = \frac{GMm}{r^2}\vec{r}, \tag{1}$$

where $\vec{r} = \overrightarrow{PO}$ and $r = |\vec{r}|$ is the magnitude of $\vec{r}$. Throughout the article, we shall use the notation of the magnitude of a vector by simply omiting the arrow over the symbol.

The energy $E$ is the sum of the kinetic and the potential energies:

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}. \tag{2}$$

The sign of $E$ is crcuial. For negative energy the trajectory is an ellipse, for positive energy the trajectory is (one branch) of a hyperbola and if the energy is zero then the trajectory is a parabola. We now set out to explain how these trajectories can be constructed from the data given in the starting question.

**Theorem 1.1.** *Let $O$ be the location of a force which attracts proportionally to the inverse of the square of the distance* (1) *and $P$ the position of a mass $m$ with inital velocity $\vec{v}$. For illustration see Figure 1.*

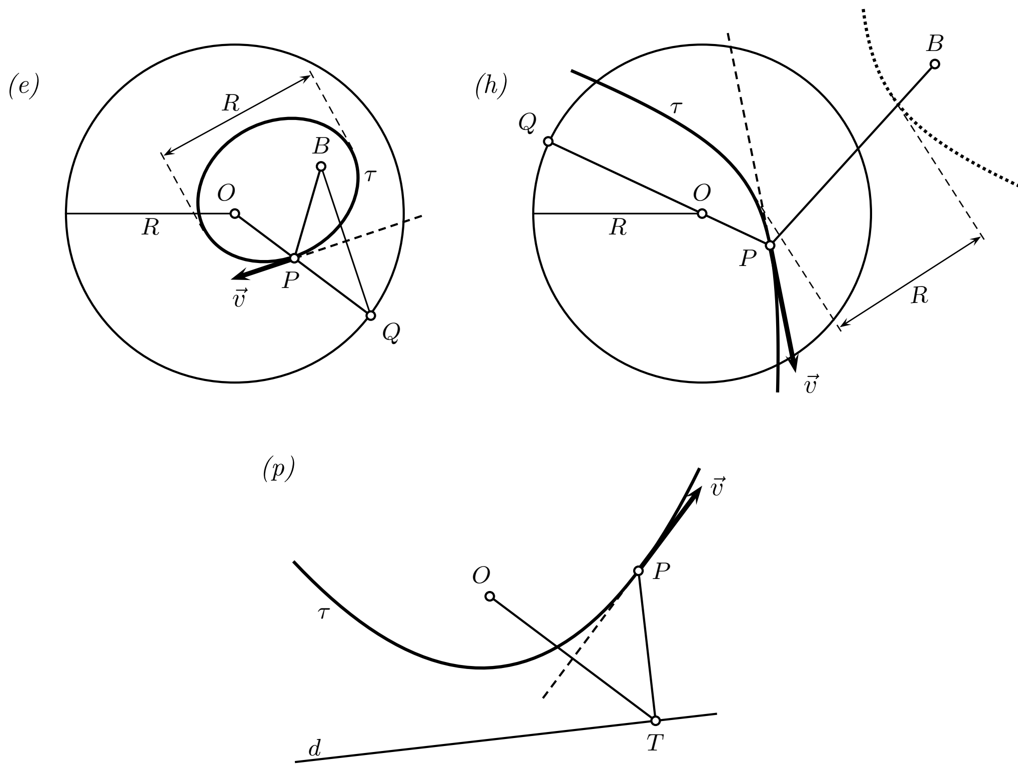


FIGURE 1. The construction of the trajectory $\tau$ according to the sign of the energy: (e) if $E < 0$, the trajectory $\tau$ is an ellipse, (h) if $E > 0$, the trajectory $\tau$ is one branch of a hyperbola, (p) if $E = 0$, the trajectory $\tau$ is a parabola.

(e) *If the energy* (2) *is negative, then let $R$ be the maximal distance $m$ may reach (in case $\vec{v}$ is parallel to $OP$), that is*
$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{R}.$$
*Let $c$ be the circle with center $O$ and radius $R$ and let $Q$ be the intersection of $c$ with the ray $OP$ starting in $O$. Further let $B$ be the reflection of $Q$ in $\vec{v}$ (that is, the line passing through $P$ and parallel to $\vec{v}$). The trajectory of $m$ is the ellipse with foci $O$ and $B$ and major axis $R$.*

(h) *If the energy $E$ is positive, then let $R$ be such that*
$$E = \frac{GMm}{R}.$$
*Let $c$ be the circle with center $O$ and radius $R$ and let $Q$ be the intersection of $c$ with the ray $PO$ starting in $P$. Further let $B$ be the reflection of $Q$ in*

*$\vec{v}$. The trajectory of $m$ is the arm of the hyperbola with foci $O$ and $B$ and major axis $R$ which is closer to $O$.*

(p) *If the energy is zero, then let $T$ be the reflection of $O$ in $\vec{v}$ and let $d$ be the line through $T$ which is perpendicular to $PT$. The trajectory of $m$ is the parabola with focus $O$ and directrix $d$.*

*Remark* 1.2. The key ingredient to prove Theorem 1.1 is that in all three cases (e), (h) and (p), the velocity $\vec{v'}$, in every point $P'$, on the trajectory is perpendicular to $\vec{q'} = \overrightarrow{BQ'}$ and $v'$ is proportional to $q'$. Here, $Q'$ is obtained from $P'$ as $Q$ is from $P$ and point $B$ will not change. In the case of zero energy, $B$ is the vertex of the parabola, $c$ is the circle with center $O$ and radius $|OB|$ and finally $Q$ is the intersection of $c$ with ray $PO$ starting in $P$, see Figure 2.

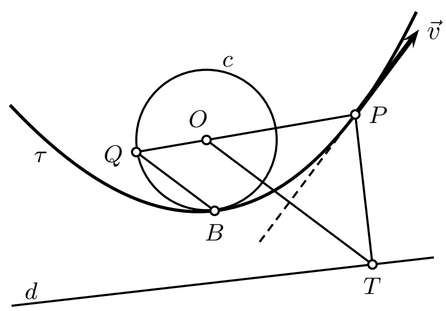


FIGURE 2. The location of the points $B$, $Q$ and the circle $c$ in case (p).

The above results were found when we studied the projectile motion. Since the results concering the latter are interesting on their own and can be proved using similar techniques, we treat them in parallel. Thus, we consider the case of a homogeneous force field, that is $\vec{F} = m\vec{g}$ holds at any position for a constant $\vec{g}$. For the sake of simplicity, we assume $\vec{g}$ to be "vertical" and "directed downwards".

The mass $m$ at position $O$ has initial velocity $\vec{v}_0$. Decomposing the initial velocity into a horizontal and a vertical part it is easy to see that the trajectories are parabolas with vertical axis.

We consider the case where $v_0$ is constant but the direction of $\vec{v}_0$ is variable, like a canon shooting under different elevation angles. Let $h$ be such that

$$\frac{1}{2}mv_0^2 = Gmh.$$

Let $d$ be a horizontal line at height $h$ above $O$. By setting the potential energy to be $E_{\text{pot}} = Gmy$, where $y$ is the vertical distance below $d$, we force $E_{\text{pot}}$ to be zero at $d$. The following result describes further properties of the possible trajectories.

**Theorem 1.3.** *We assume the force field to be homgeneous. A mass $m$ at position $O$ has initial velocity $\vec{v}_0$. Let $h$ be such that $\frac{1}{2}mv_0^2 = Gmh$ and let $d$ be a horizontal line at distance $h$ above $O$. Let $O'$ be the point on $d$ closest to $O$, that is, directly above $O$. Let $F$ be the reflection of $O'$ on $\vec{v}_0$ and $\tau$ the parabola with focus $F$ and directrix $d$. Then $\tau$ is the trajectory of the mass $m$, see Figure 3. Let $B$ be the vertex of $\tau$ and $c_\tau$ the circle with center $F$ and radius $|FB|$. For any point $P$ on $\tau$,* *the velocity $\vec{v}$ is proportional to $q = |BK|$, namely $v = \sqrt{\frac{2g}{h}} q$, where $K$ is obtained as the intersection of $d$ with the line $LB$ and $L$ is the intersection of $PF$ with $c_\tau$ which is farthest from $P$.*

*For varying directions of $\vec{v}_0$ but fixed magnitude $v_0$ the following assertions hold (see Figure 4 for illustration).*

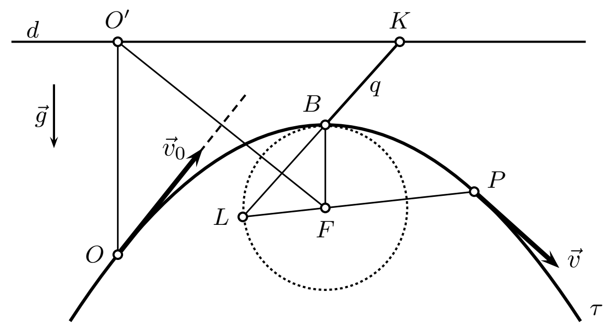


FIGURE 3. Parabolic trajectories in a homogeneous force field (indicated by the aceleration $\vec{g}$).

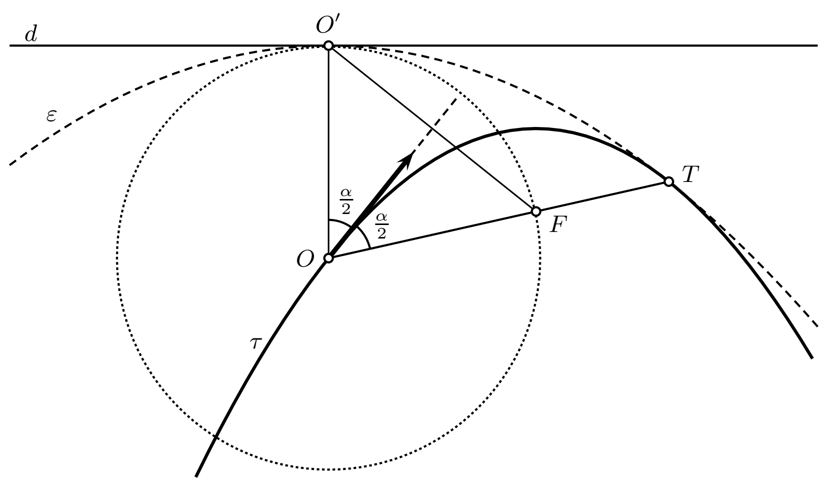


FIGURE 4. The envelope $\varepsilon$ of all possible trajectories where the direction of $\vec{v}_0$ varies but the magnitude $v_0$ remains constant.

(i) *The focus $F$ of the trajectory $\tau$ lies on a circle with center $O$ and radius $h$ and $d$ is the directrix of $\tau$.*

(ii) *The envelope $\varepsilon$ of the various parabolic trajectories is a parabola with focus $P$ and vertex $B$.*

(iii) *The maximal distance possible to reach in direction $\alpha$ (measured from the vertical) is the point where the trajectory meets the envelope $\varepsilon$ and it is attained when the initial velocity $\vec{v}_0$ is the angular bisector of $\alpha$.*

Property (iii) is well known in the special acse, where $\alpha = 90°$ as the rule that –in vacuum and on flat ground– the maximum range of a projectile is attained by launching it with an elevation of $45°$.

The rest of the article is organized as follows. We first prove, in Section 2, some technical results on conic sections. In Section 3 we show that for the central force field, when the velocity is proportional to $|O'Q|$ and perpendicular to $O'Q$, the energy and the angular momentum are preserved. In Section 4 we show how this implies that the described curve is indeed the trajectory, prove Kepler's laws and describe the hodograph. In Section 5 we treat the projectile motion.

## 2. Some geometric properties of conic sections

We prove three rather technical results about conic sections, one about ellipses, one about hyperbolas and a third one about parabolas.

**Lemma 2.1.** *Let $O$ be the center of a circle $c$ with radius $R$ and let $B$ be any point in the interior of $c$ different from $O$. Let $\tau$ be the ellipse with the foci $O$ and $B$, and major axis equal to $R$. Let $P$ be any point on $\tau$ and let $Q$ be the intersection of $c$ with the ray $OP$ (starting in $O$). We define $f = |OB|$, the focal distance and further, $r = |OP|$ and $q = |BQ|$. Then the following equalities hold:*

$$q^2 = \frac{(R^2 - f^2)(R - r)}{r} \tag{3}$$

$$rq \sin\theta = \frac{R^2 - f^2}{2}, \tag{4}$$

*where $\theta$ is the angle formed by the line $OQ$ and the tangent to $e$ at $P$.*

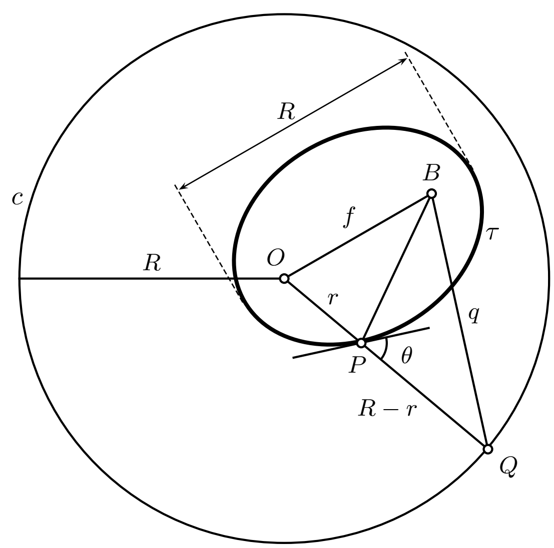


Figure 5. Illustration for Lemma 2.1.

*Proof.* Since the major axis of the ellipse $\tau$ equals $R$ we have $|OP| + |PB| = R$ and therefore $|PB| = |PQ| = R - r$. By the reflection property of the ellipse the tangent $t$ to $\tau$ at $P$ is the bisector of the segment $BQ$ and therefore

$$\sin\theta = \frac{q}{2(R - r)}. \tag{5}$$

The law of cosine in the triangle $\Delta OQB$ yields

$$f^2 = R^2 + q^2 - 2Rq\cos\varphi, \tag{6}$$

where $\varphi = \measuredangle OQB$. We observe that $\sin\theta = \cos\varphi$ and substitute (5) into (6) to get:

$$f^2 - R^2 = q^2 - \frac{Rq^2}{R-r} = \frac{-rq^2}{R-r}, \tag{7}$$

from which (3) follows at once.

To see (4) we multiply (5) by $rq$ and substitute $q^2$ using (3):

$$rq\sin\theta = \frac{r}{2(R-r)}q^2 = \frac{r(R^2-f^2)(R-r)}{2(R-r)r} = \frac{R^2-f^2}{2}$$

This concludes the proof of Lemma 2.1 □

**Lemma 2.2.** *Let $O$ be the center of a circle $c$ with radius $R$ and let $B$ be any point in the exterior of $c$. Let $\tau$ be the hyperbola with focii $O$ and $B$, and major axis equal to $R$. Let $P$ be any point on $\tau$ and $Q$ be the intersection of $c$ with the ray $PO$ (starting at $P$). We define $f$ to be the focal distance $|OB|$ and further $r = |OP|$ and $q = |BQ|$. Then the following equalities hold:*

$$q^2 = \frac{(f^2-R^2)(R+r)}{r} \tag{8}$$

$$rq\sin\theta = \frac{f^2-R^2}{2}, \tag{9}$$

*where $\theta$ is the angle formed by the line $OQ$ and the tangent to $\tau$ at $P$.*

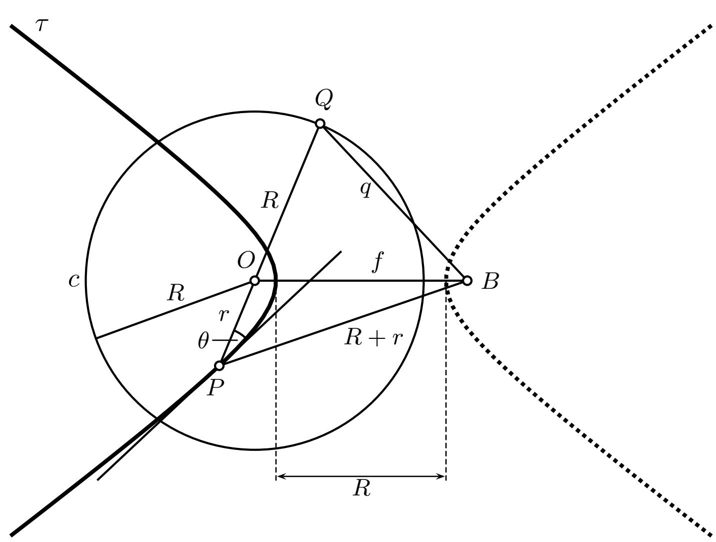


FIGURE 6. Illustration for Lemma 2.2.

*Proof.* Since the major axis of $\tau$ is $R$, we have $|PB| - |PO| = R$. Therefore $|PB| = R + r = |PO| + |OQ| = |PQ|$. The reflection property of the hyperbola implies that the tangent at $P$ to $\tau$ is the angular bisector of the triangle $\Delta PBQ$. The rest of the proof follows now in complete analogy to the proof of Lemma 2.1. $\square$

**Lemma 2.3.** *Let $c$ be a cicrle with center $O$ and radius $R$ and let $B$ be a point on $c$. Then let $\tau$ be the parabola with focus $O$ and vertex $B$. Let $P$ be any point of $\tau$ and let $Q$ be the intersection of $c$ with the line $PO$ which is farthest from $P$. Let $r = |OP|$ and $q = |QB|$. Then*

$$q^2 = \frac{4R^3}{r} \tag{10}$$

$$rq\sin\theta = \frac{R^2}{2}, \tag{11}$$

$$|BK|^2 = Rr \tag{12}$$

*where $\theta$ is the angle formed by the line $OP$ and the tangent to $p$ at $P$.*

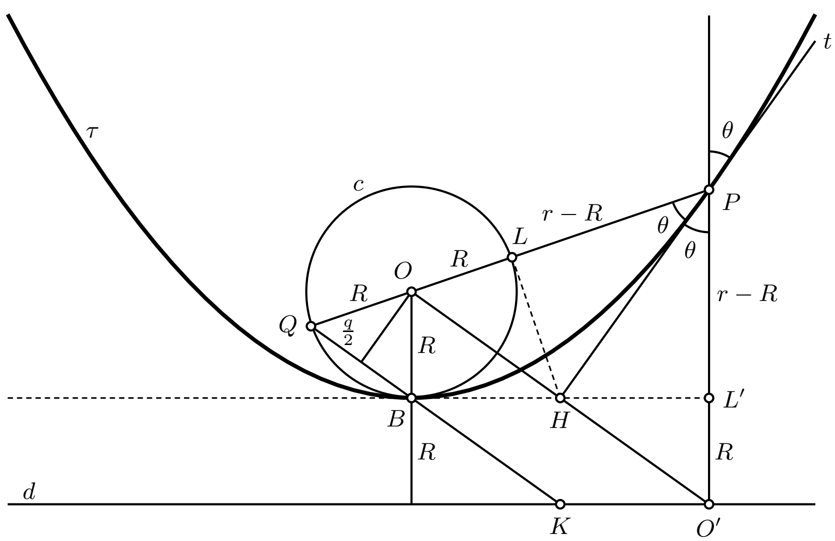


FIGURE 7. Illustration for Lemma 2.3.

*Proof.* The directrix is the perpendicular to the symmetry axis $OB$ at distance $R$ from the vertex $B$. The point $P$ has the same distance $r$ from the focus $O$ than from the directrix $d$, thus $|PO| = |PO'|$. By the reflection property of the parabola, parallel light to the symmetry axis is reflected to the focus. Therefore the tangent $t$ is the angular bisector at $P$ of the triangle $\Delta OPO'$. Denote by $H$ the midpoint of $OC$ and by $L$ (resp. by $L'$) the foot of the height of the rectangular triangle $\Delta PHO$ (resp. $\Delta PHO'$).

Since $PO'$ is parallel to $OB$, we have $\sphericalangle OPO' = \sphericalangle QOB$. By similarity we have $\frac{q}{2R} = \frac{|OH|}{r}$ and therefore

$$q^2 = 4R^2\frac{|OH|^2}{r^2} \overset{(*)}{=} 4R^2\frac{rR}{r^2} = \frac{4R^3}{r},$$

where equation $(*)$ follows from the fact that the square of a leg of a right triangle equals the product of the hypothenuse with it's adjacent part of the hypothenuse cut off by the height, so: $|OH|^2 = rR$.

Now $r\sin(\theta) = |OH|$ and by similarity we have $\frac{|OH|}{R} = \frac{2R}{q}$. Hence

$$rq\sin(\theta) = |OH|q = \frac{|OH|}{R}Rq = \frac{2R}{q}Rq = 2R^2,$$

which shows (11) whereas (12) follows from the fact that $|BK|^2 = |HO'|^2 = |OH|^2 = Rr$. □

## 3. Conservation of energy and angular momentum

We study the situation, where a mass $m$ is attracted in a central force field which is proportional to the inverse of the square of the distance. We denote by $\vec{v}$ the velocity of this mass and by $v$ its magnitude. Hence the kinetic energy of the mass $m$ equals $\frac{1}{2}mv^2$. To simplify our treatment, we assume the mass $m$ to be really small compared to $M$, so the potential energy is equal to $-\frac{GMm}{r}$, where $r$ is the distance between the two masses and $G$ is the universal gravitational constant. The total energy of the particle $m$ is therefore

$$E = \frac{mv^2}{2} - \frac{GMm}{r}.$$

The orbit of the mass $m$ depends crucially on the sign of $E$: if $E$ is negative then it is an ellipse; if $E = 0$, then it is a parabola; and if $E > 0$ then, it is a hyperbola. We study first the most interesting case where the energy is negative.

**Proposition 3.1.** *With the notations of Section 2, we have that if the mass $m$ moves along a curve $\tau$ with a velocity $\vec{v}$ which is always perpendicular to $\overrightarrow{QB}$ such that its magnitude $v$ is proportional to $q = |QB|$, namely $v = \lambda q$ with*

$$\lambda = \begin{cases} \sqrt{\frac{2GM}{R(R^2-f^2)}}, & \text{if } E < 0, \\ \sqrt{\frac{2GM}{R(f^2-R^2)}}, & \text{if } E > 0, \\ \sqrt{\frac{GM}{2R^3}}, & \text{if } E = 0, \end{cases} \tag{13}$$

*then the energy and the angular momentum are preserved.*

*Proof.* We first consider the case where the energy is negative. We then have

$$\frac{1}{2}mv^2 = \frac{GMm}{R(R^2-f^2)}q^2 = GMm\frac{R-r}{Rr} = GMm\left(\frac{1}{r} - \frac{1}{R}\right),$$

where the first equation follows from (13) and the second by Lemma 2.1. Hence

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{R},$$

that is, the energy $E$ is constant. The angular momentum is clearly perpendicular to the plane of the ellipse $e$ and its magnitude $L$ is such that $L = mrv\sin(\theta)$. Since $v$ is proportional to $q$ and $qr\sin(\theta)$ is constant we get that $L$ is constant.

The proof for the other two cases are analogous. □

## 4. Conclusions

4.1. **Newton's second law.** The next result shows that the preservation of energy and angular momentum imply that Newton's second law is also satisfied.

**Theorem 4.1.** *With the hypothesis of Proposition 3.1, we have that the movement of $m$ along $\tau$ satisfies Newton's second law:*

$$\vec{F} = m\vec{a}, \tag{14}$$

*where the force $\vec{F}$ is given by $\vec{F} = -\frac{GMm}{r^3}\vec{r}$ and $\vec{a}$ is the acceleration $\vec{a} = \frac{d^2}{dt^2}\vec{r}$.*

*Proof.* We may assume that the velocity $\vec{v}$ and postion vector $\vec{r}$ are parametrized by time $t$ such that $\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{v}$ (see Appendix, Section 6 for a brief explanation of this).

Since the angular momentum $\vec{L}$ is always perpendicular to the plane of the ellipse $e$ and by Proposition 3.1, it is also constant in magnitude, we have $\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = \vec{0}$. Since $\vec{L} = \vec{r} \times (m\vec{v})$ we get

$$\vec{0} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = m\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \times \vec{v} + \vec{r} \times \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \vec{r} \times \vec{a},$$

where the last equality follows from the fact that $\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{v}$ (by Lemma 6.1), $\vec{v} \times \vec{v} = 0$ and $\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \vec{a}$. This shows that at each position, the aceleration $\vec{a}$ is parallel to $\vec{r}$.

Since the energy is also constant we have that $\frac{\mathrm{d}E}{\mathrm{d}t} = 0$ and hence

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m\vec{v}\cdot\vec{v}}{2} - \frac{GMm}{(\vec{r}\cdot\vec{r})^{\frac{1}{2}}}\right) = m\vec{v}\cdot\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} + \frac{GMm}{(\vec{r}\cdot\vec{r})^{\frac{3}{2}}}\vec{r}\cdot\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = m\vec{v}\cdot\vec{a} + \frac{GMm}{r^3}\vec{r}\cdot\vec{v}$$

Therefore

$$0 = \vec{v}\cdot\left(m\vec{a} + \frac{GMm}{r^3}\vec{r}\right) = \vec{v}\cdot(m\vec{a} - \vec{F})$$

Since $v$ is proportional to $q$, a segment which never vanishes, we have $v \neq 0$ and therefore (14) holds unless $\vec{v}$ is perpendicular to $m\vec{a} - \vec{F}$. Note that $\vec{a}$ and $\vec{r}$ (thus also $\vec{F}$) are parallel. Consequently, if the trajectory is not circular, (14) holds in all but a finite number of points along the curve and thus, by continuity, on all points.

It remains to consider the case, where the trajectory is circular. In this case the focal distance vainshes, that is $f = 0$, further $q = R$ and the distance $r$ to the origin $O$ is constant $r = \frac{R}{2}$. By Propostion 3.1 $v = \sqrt{\frac{2GM}{R^3}}q$ and therefore

$$v^2 = \frac{2GM}{R^3}q^2 \overset{(3)}{=} \frac{2GM(R-r)}{Rr} = \frac{2GM}{R} = \frac{GM}{r}$$

Since the acceleration $a$ in uniform circular motion satisfies $a = \frac{v^2}{r}$ we get

$$ma = m\frac{v^2}{r} = \frac{GMm}{r^2} = F$$

and hence the result. □

Therefore, according to the sign of the energy $E$, the ellipse $e$, hyperbola $h$ and parabola $p$ are the trajectories of the mass $m$ moving in the central force field which is proportional to the inverse of the square distance.

4.2. **Proof of Remark 1.2.** Indeed, since the trajectory is precisely the conic section described geometrically, Remark 1.2 follows at once from Proposition 3.1.

4.3. **Kepler's laws.** We assume the notations of the previous sections.

**Proposition 4.2.** *Suppose that a mass $m$ is attracted by a central force, located at $O$, which is proportional to the inverse of the square distance. Then the following assertions hold.*

(a) *The trajectory of the orbit is (part of) a conic section: either an ellipse, a parabola or one branch of a hyperbola. In each case $O$ is a focus of the trajectory.*
(b) *The position vector $\vec{r}$ sweeps out equal areas during equal intervals of time.*
(c) *The square of the orbital period $T$ of an elliptic trajectory is proportional to the cube of the major axis $R$, more precisely,*

$$T^2 = \frac{\pi^2}{2GM} R^3.$$

*Proof.* Part (a) follows directly from the fact that the curves constructed in Theorem 1.1 –which are conic sections– are the trajectories (and all the possible trajectories). However, parabola and hyperbola correspond to orbits of an object passing only once through each point. Hence the orbits of revolving objects must be elliptic.

Part (b) is an immediate consequence of the preservation of the angular momentum $\vec{L}$. The area swept by $\vec{r}$ in the interval $[t_0, t_1]$ is given by

$$A_{t_0}^{t_1} = \int_{t_0}^{t_1} \frac{1}{2} r(t) v(t) \sin(\theta) \mathrm{d}t = \int_{t_0}^{t_1} \frac{L}{2m} \mathrm{d}t = \frac{L}{2m}(t_1 - t_0). \tag{15}$$

This shows that the area is proportional to the elapsing time.

For part (c), we use first (15) to calculate the oribtal period $T$:

$$T = \frac{2mA}{L},$$

where $A$ is the area of the ellipse $e$. We have

$$A = \frac{\pi}{4} R \sqrt{R^2 - f^2}, \tag{16}$$

since $e$ has minor axis $\sqrt{R^2 - f^2}$. The magnitude $L$ of the angular momentum can be calculated following the ideas of the proof of Proposition 3.1:

$$L = rmv \sin(\theta) \overset{(13)}{=} m \sqrt{\frac{2GM}{R(R^2 - f^2)}} qr \sin(\theta) \overset{(4)}{=} m \sqrt{\frac{2GM(R^2 - f^2)}{2R}}. \tag{17}$$

Putting all together we obtain

$$T = \frac{2mA}{L} \overset{(16)}{=} \frac{\pi m R \sqrt{R^2 - f^2}}{2L} \overset{(17)}{=} \frac{\pi R \sqrt{R}}{\sqrt{2GM}}, \tag{18}$$

and hence (c) follows by squaring (18). $\square$

4.4. **The Hodograph.** The hodograph is by definition the curve formed by the endpoints of the velocity vectors $\vec{v}$, when translated to a fixed starting point.

**Proposition 4.3.** *The hodograph is always part of a cicle. If $\tau$ denotes the trajectory and $H$ the hodograph then we have the following assertions.*

(a) *If $\tau$ is an ellipse, then $H$ is a full circle.*
(b) *If $\tau$ is a parabola, then $H$ is an open arc obtained from a circle by removing a single point.*
(c) *If $\tau$ is a hyperbola, then $H$ is an open arc obtained from a circle by removing a closed arc segment.*

*Proof.* Since $\vec{v}$ is in each case proportional to $\vec{q} = \overrightarrow{BQ}$ and always perpendicular, we may as well take $\vec{q}$ to investigate the form of the hodograph $H$. Let $H'$ be the curve formed by the endpoints of all vectors $\vec{q}$. In each case $\vec{q}$ starts in $B$ and ends in the cicle $c$. Hence $H'$ is always a subset of $c$. If $\tau$ is an ellipse, $B$ is an interior point of $c$ and $H' = c$ follows. If $\tau$ is a parabola, then $B$ lies on $c$ and $H' = c \setminus \{B\}$, that is, the whole circle $c$ except for the point $B$. If $\tau$ is a hyperbola then $H'$ is an open arc delimited by the tangent points of $B$ to $c$, since for $r \to \infty$ we obtain $q^2 = \frac{(f^2-R^2)(R+r)}{r} \xrightarrow{r\to\infty} f^2 - R^2$ and thus the triangle $\Delta OBQ$ approaches a rightangled one (see Figure 8 for illustration). Consequently the line $BQ$ approaches the tangent at $c$ through $B$. $\square$

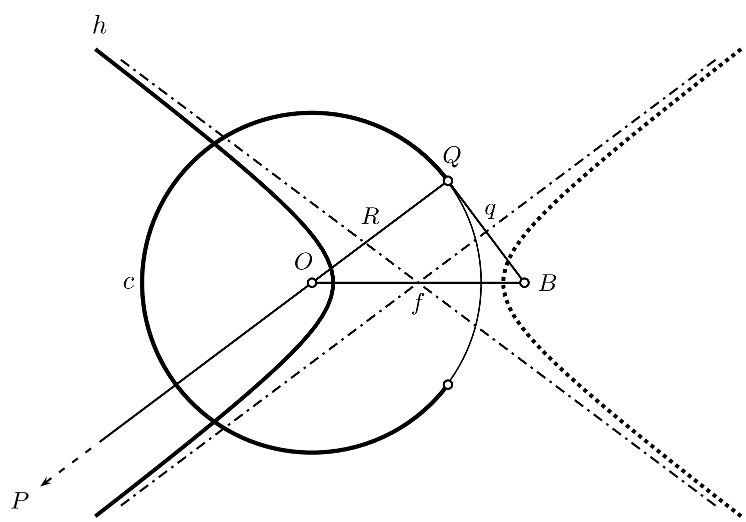


FIGURE 8. Limit case for $P$ infinitely far away.

## 5. Projectile motion

We finally give the proof of Theorem 1.3. By Lemma 2.3 we know that the tangent at point $P$ is perpendicular to $BL$. Since $v$ is assumed to be proportional to $|BL|$ the aceleration $\vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$ is vertical, due to the fact that the variation of $\overrightarrow{BK}$ is horizontal. Note that again we assumed $\vec{v}$ and $\vec{r}$ to be parametrized by time in such a way that $\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{v}$ (see Appendix, Section 6).

The energy is $E = \frac{mv^2}{2} - mgy$, where $y$ is the distance from $P$ to $d$. Using $v = \sqrt{\frac{2g}{h}}$ and (12) of Lemma 2.3, where $R = h$, we have

$$E = \frac{m\frac{2g}{h}q^2}{2} - mgy = mg\left(\frac{hr}{h} - y\right) = 0,$$

where the last equation follows from $y = r$, since $P$ has the same distance from $F$ than from $d$. Thus the energy is constant.

We decompose the velocity into a horizontal and a vertical part $\vec{v} = \vec{v}_{\mathrm{h}} + \vec{v}_{\mathrm{v}}$. Thus

$$E = \frac{mv_{\mathrm{h}}^2}{2} + \frac{mv_{\mathrm{v}}^2}{2} - mgy$$

Using that the aceleration is vertical we have $\frac{\mathrm{d}v_{\mathrm{h}}}{\mathrm{d}t} = 0$ and therefore

$$\begin{aligned} 0 = \frac{\mathrm{d}E}{\mathrm{d}t} &= mv_{\mathrm{h}}\frac{\mathrm{d}v_{\mathrm{h}}}{\mathrm{d}t} + mv_{\mathrm{v}}\frac{\mathrm{d}v_{\mathrm{v}}}{\mathrm{d}t} - mg\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= m\left(v_{\mathrm{v}}a - gv_{\mathrm{v}}\right) \end{aligned}$$

showing that $a = g$. Thus Newton's second law is satisfied and the parabola is indeed the trajectory. Now part (i) of Theorem 1.3 follows immediatley from the construction.

Let $\varepsilon$ be the parabola with focus $O$ and vertex $O'$, see Figure 9 for illustration. Let $Q$ be any point of $\tau$ and $Q'$ (resp. $Q''$) the intersection of the vertical line through $Q$ with $d$ (resp. with $d_\varepsilon$).

Then we have

$$|OQ| \leq |QF| + |FO| = |QQ'| + h = |QQ''|$$

with equality if and only if $F$ lies on $OQ$. This shows that $\varepsilon$ touches $\tau$ in a single point, namely the intersection $T$ of $OF$ with $\tau$. Hence $\varepsilon$ is the envelope of all possible trajectories for varying directions of $\vec{v}_0$ and constant magnitude $v_0$, which proves part (ii) of Theorem 1.3. Furthermore, the farthest point from $O$ which may be reached in direction $\alpha$ (measured from the vertical) is obtained by the trajectory for which the initial velocity $\vec{v}_0$ is the angular bisector of $\alpha$, since $\vec{v}_0$ is the angular bisector of $\sphericalangle O'OF$. Hence property (iii) is also shown and we have completed the proof of Theorem 1.3.

## 6. Appendix

In each of the cases, we have given the velocity at each position along a path, that is, we have the velocity given as a function $\vec{v}(\vec{\rho})$ of the position vector $\vec{\rho}$. If we parametrize the path by the arc length $s$ to the initial position, then we get the function $\vec{\rho} = \vec{\rho}(s)$.

If a particle moves along the path in such a way that at each position $\vec{\rho}(s)$ the velocity is $\vec{v}(\vec{\rho}(s))$, then we obtain that the time as a function of the arc length is given by

$$t(s) = \int_0^s \frac{\mathrm{d}\sigma}{v(\vec{\rho}(\sigma))}. \tag{19}$$

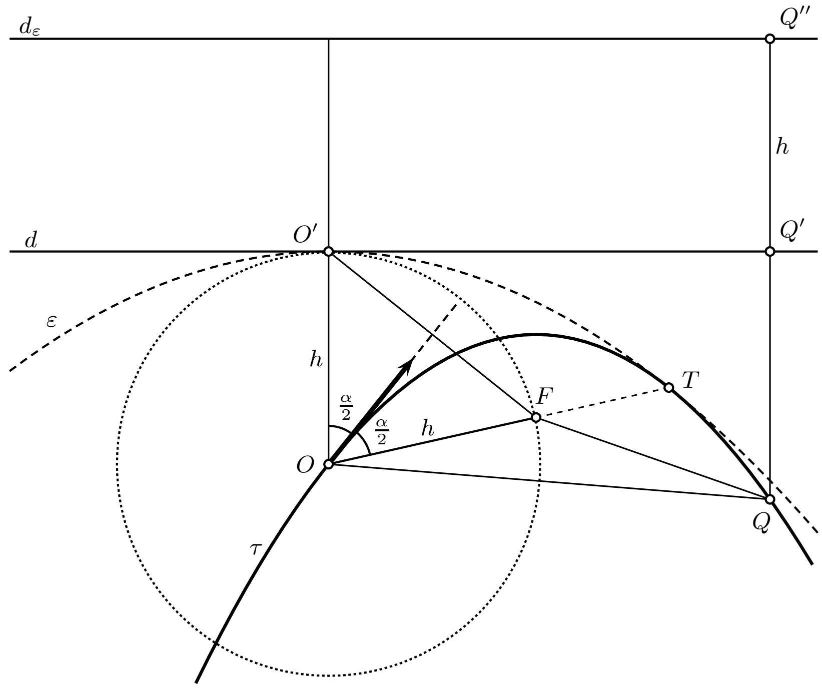


FIGURE 9. The envelope $\varepsilon$ of all possible trajectories where the direction of $\vec{v}_0$ varies but the magnitude $v_0$ remains constant.

Indeed, this is easily seen using a limit process: we subdivide the arc between the initial position $\sigma = 0$ and the end position $\sigma = s$ into $n$ parts $s_0 = 0, s_1, \ldots, s_n = s$. Let $t_i$ be the time at which the particle passes position $\vec{\rho}(s_i)$. Then we have that the arc difference $s_{i+1} - s_i$ is approximately equal to

$$s_{i+1} - s_i \approx (t_{i+1} - t_i) \cdot v(\vec{\rho}(s_i)).$$

Hence

$$t(s) = t_n = \sum_{i=1}^{n} t_{i+1} - t_i \approx \sum_{i=0}^{n} \frac{s_{i+1} - s_i}{v(\vec{\rho}(s_i))}$$

yielding equaltity (19) for $n \to \infty$.

Since the velocity is nowhere declared zero, we get that the time is monotonously increasing with the arc length. Hence there exists the inverse function $s(t)$ of the arc length as a function of the time. Now we set

$$\vec{r}(t) = \vec{\rho}(s(t)) \quad \text{and} \quad \vec{v}(t) = \vec{v}(s(t)).$$

**Lemma 6.1.** *With the above notation and definitions, we have* $\frac{\mathrm{d}\vec{r}(t)}{\mathrm{d}t} = \vec{v}(t)$.

*Proof.* We have the following equalities:

$$\frac{\mathrm{d}\vec{r}(t)}{\mathrm{d}t} = \frac{\mathrm{d}\vec{r}(s(t))}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}\vec{r}(s)}{\mathrm{d}s} v(t) = \vec{v}(t),$$

where the last equality follows from the fact that $\frac{\mathrm{d}\vec{r}(s)}{\mathrm{d}s}$ is a unit vector tangent to the path and $\vec{v}$ has length $v$ and is also tangent to the path. □

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